Manifolds with Boundary

- The model for manifolds with boundary are the closed n-dimensional **upper half-space** $\mathbb{R}^n_+ \subset \mathbb{R}^n$, defined as

$$\mathbb{R}^n_+ = \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^n \geq 0\}.$$

Use $\mathrm{Int}\mathbb{R}^n_+$ and $\partial\mathbb{R}^n_+$ to denote the interior and boundary of $\mathbb{R}^n_+$:

$$\begin{aligned} \mathrm{Int}\,\mathbb{R}^n_+ =& \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^n > 0\} \\ \partial\mathbb{R}^n_+ =& \{(x^1, \cdots, x^n) \in \mathbb{R}^n : x^n = 0\}. \end{aligned}$$

Definition. *An n-dimensional* **topological manifold with boundary** *is a second countable Hausdorff space M, equipped with coordinate charts (U_i, φ_i) such that*
(i) the U_i's form an open covering of M;
(ii) φ_i is a homeomorphism from U_i to a relatively open subset of $\mathbb{R}^n_+$.
- *We call (U, φ) an* **interior chart** *if $\varphi(U) \subset \mathrm{Int}\,\mathbb{R}^n_+$, and a* **boundary chart** *if $\varphi(U) \cap \partial\mathbb{R}^n_+ \neq \emptyset$.*

Definition. *If U is an open subset of $\mathbb{R}^n_+$, a map $F : U \to \mathbb{R}^k$ is smooth if for each $x \in U$, there exists*

(1) *an open subset of ∂R^n_+ and*
(2) *a smooth map $\widetilde{F} : V \to \mathbb{R}^k$ with $\widetilde{F}\big|_{V \cap \mathbb{R}^n_+} = F$.*

Example. Let $\mathbb{B}^2 \subset \mathbb{R}^2$ be the open unit disk, let

$$U = \mathbb{B}^2 \cap \mathbb{R}^n_+.$$

(1) Define $f : U \to \mathbb{R}$ by

$$f(x, y) = \sqrt{1 - x^2 - y^2}.$$

Because f extends smoothly to $\mathbb{B}^2$ (by the same formula), f is a smooth function on U.

(2) Define $g : U \to \mathbb{R}$ by

$$g(x, y) = \sqrt{y}.$$

Although g is continuous on U and smooth in $U \cap \mathrm{Int}\,\mathbb{R}^n_+$, it has no extension to any neighborhood of the origin in $\mathbb{R}^2$, (because $\frac{\partial g}{\partial y} \to \infty$ as $y \to 0$.) Thus g is not smooth on U.

Definition. *An n-dimensional* **smooth manifold with boundary** *is topological manifold with boundary M, equipped with coordinate charts (U_i, φ_i) such that whenerver $U_i \cap U_j \neq \emptyset$ for any pair of indices $i \neq j$, the transition function*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a smooth diffeomorphism.

Orientation of Hypersurfaces

- If M is an oriented maifold and N is a submanifold of M, N may not inherit an orientation from M, even if N is embedded.
- Clearly, it is not sufficient to restrict an orientation form from M to N, since the restriction of an n-form to a manifold of lower dimension must necessarily be zero.

Example. The Moöbius band is not orientable, even though it can be embedded in $\mathbb{R}^3$.

- With one extra piece of information (a certain kind of vector field along the hypersurface), we can use an orientation on M to induce an orientation on any hypersurface $S \subset M$.

Definition. *Let V be a finite-dimensional vector space, and let $X \in V$. We define a linear map*

$$i_X : \Lambda^k(V) \to \Lambda^{k-1}(V),$$

called **interior multiplication** *or* **contraction** *with X, by*

$$i_X\omega(Y_1, \cdots, Y_{k-1}) = \omega(X, Y_1, \cdots, Y_{k-1}).$$

In other words, $i_X\omega$ is obtained from ω by inserting X into the first slot.
— *By convention, we set $i_X\omega$ to be zero when ω is a 0-covector (i.e. a number).*
— *Another common notation is*

$$X \rfloor \omega = i_X\omega.$$

Definition. *On a smooth manifold M, interior multiplication extends naturally to vector fields and differential forms, simply by letting it act pointwise: If $X \in \mathcal{T}(M)$ and $\omega \in \mathcal{A}^k(M)$, define a $(k-1)$-form $X \lrcorner \omega = i_X\omega$ by*

$$(X \rfloor \omega)_p = X_p \lrcorner \omega_p.$$

Definition. *Suppose M is a smooth manifold and $S \subset M$ is a submanifold (immersed or embedded).*

- *A vector field* **along** *S is a continuous map $X : S \to TM$ with the property that $X_p \in T_pM$ for each $p \in S$.*
- ⊙ *A vector $X_p \in T_pM$ at some point $p \in S$ is said to be* **transverse** *to S if T_pM is spanned by X_p and T_pS.*
- ⊙ *A vector field X along S is* **transverse** *to S if X_p is transverse to S at each $p \in S$.*

Proposition 1. *Suppose M is an oriented smooth n-manifold, S is an immersed hypersurface in M, and X is a* **transvese** *vector field along S.*
Then S has a unique orientation such that for each $p \in S$, $(E_1, \cdots, E_{n-1})$ is an oriented basis for T_pS iff $(X_p, E_1, \cdots, E_{n-1})$ is an oriented basis for T_pM.

- *If Ω is an orientation form for M, then $(X \lrcorner \Omega)\Big|_S$ is an orientation form for S w.r.t. this orientation.*

Proof. Let Ω be an orientation for M. Then $\omega = (X \lrcorner \Omega)\big|_S$ is an $(n-1)$-form on S.

— **To show that $X \lrcorner \Omega$ is an orientation form for** S, it suffices to **claim: ω never vanishes.**

– Indeed, given any basis $(E_1, \cdots, E_{n-1})$ for T_pS, the fact that X is transverse to S implies that $(X, E_1, \cdots, E_{n-1})$ is a basis for T_pM.

– The fact that Ω is nonvanishing implies that

$$\omega_p(E_1, \cdots, E_{n-1}) = X \lrcorner \Omega(E_1, \cdots, E_{n-1}) = \Omega(X, E_1, \cdots, E_{n-1}) \neq 0.$$

Since $\omega_p(E_1, \cdots, E_n) > 0$ iff $\Omega_p(X_p, E_1, \cdots, E_n) > 0$, the orientation determined by ω is the one defined in the statement of the proposition. $\square$

Example. Considering $\mathbb{S}^n$ as a hypersurface in $\mathbb{R}^{n+1}$, the vector field

$$X = x^i \frac{\partial}{\partial x^i}$$

along $\mathbb{S}^n$ is easily seen to be transverse, so it induces an orientation on $\mathbb{S}^n$.

Lemma 2. *Let M be an oriented smooth manifold. Suppose $S \subset M$ is a regular level set of a smooth function $f : M \to \mathbb{R}$. Then S is orientable.*

Proof. Let g be any Riemannian metric on M, and let

$$X = \operatorname{grad} f \Big|_S .$$

The hypotheses imply that X is a transverse vector field along S, so the result follows from Proposition 1. $\square$

- Interior multiplication shares two important properties with exterior differentiation: They are both **antiderivations whose square is zero**.

Lemma 3. *Let V be a finite-dimensional vector space and $X \in V$.*
(a) $i_X \circ i_X = 0$.
(b) i_X is an **antiderivation***: If ω is a k-covector and η is an ℓ-covector,*

$$i_X(\omega \wedge \eta) = (i_X \omega) \wedge \eta + (-1)^k \omega \wedge (i_X \eta).$$

Proof. (a) On k-covectors for $k \geq 2$, part (a) is immediate from the definition, because any alternating tensor gives zero when two of its arguments are identical.

(b) It suffices to consider the case in which both ω and η are wedge products of 1-covectors.
For this, it suffices to prove the following general formula for covectors $\omega^1, \cdots, \omega^k$

$$X \lrcorner (\omega^1 \wedge \cdots \wedge \omega^k) = \sum_{i=1}^{k} (-1)^{i-1} \omega^i(X) \omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \wedge \omega^k.$$

– To prove this, let us write $X_1 = X$ and apply both sides to vectors $(X_2, \cdots, X_k)$; then what we have to prove is

$$(*) \qquad (\omega^1 \wedge \cdots \wedge \omega^k)(X_1, \cdots, X_k)$$
$$= \sum_{i=1}^{k} (-1)^{i-1} \omega^i(X_1)(\omega^1 \wedge \cdots \wedge \widehat{\omega^i} \wedge \cdots \omega^k)(X_2, \cdots, X_k).$$

The left-hand side of (*) is the determinant of the matrix $\mathbb{X} = (\omega^i(X_j))$.
To simplify the right-hand side, let $\mathbb{X}^i_j$ denote the $(k-1) \times (k-1)$ minor of $\mathbb{X}$ obtained by deleting the ith row and jth column. Then the right-hand side of (*) is

$$\sum_{i=1}^{k} (-1)^{i-1} \omega^i(X_1) \det \mathbb{X}^i_1.$$

This is just the expansion of $\det \mathbb{X}$ by minors along the first column, and therefore is equal to $\det \mathbb{X}$. $\square$

Boundary Orientations

We will define a canonical orientation on the boundary of any oriented smooth manifold with boundary.

Definition. *If M is a smooth manifold with boundary, ∂M is an embedded hypersurface in M, and every point $p \in \partial M$ is in the domain of a smooth boundary chart (U, φ) such that $\varphi(U \cap \partial M)$ is the slice $\varphi(U) \cap \partial\mathbb{R}^n_+$.*
- *Let $p \in \partial M$. A vector $\nu \in T_pM$ is said to be* **inward-pointing** *if $\nu \notin T_p\partial M$ and for some $\varepsilon > 0$, there exists a smooth curve segment $\gamma : [0, \varepsilon] \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = \nu$.*
- *A vector $\nu \in T_pM$ is said to be* **outward-pointing** *if $-\nu$ is inward pointing.*

- The following lemma gives another characterization of inward-pointing vectors, which is usually much more easier to check.

Lemma 4. *Suppose M is a smooth manifold with boundary, $p \in \partial M$, and (x^i) are any smooh smooth boundary coordinates in a neigborhood of p.*
The inward-pointing vectors in T_pM are precisely those with positive x^n component, the outward-pointing ones are those with negative x^n-component.

Lemma 5. *If M is any smooth manifold with boundary, there is a smooth outward pointing vector field along ∂M.*

Proof. Cover a neighborhood of ∂M by smooth boundary charts $\{(U_\alpha, \varphi_\alpha)\}$.
— In each such chart

$$\nu_\alpha = -\frac{\partial}{\partial x^n}\Big|_{\partial M \cap U_\alpha}$$

is a smooth vector field along $\partial M \cap U_\alpha$, which is outward-pointing by Lemma 4.
— Let $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to the cover $\{\partial M \cap U_\alpha\}$ of ∂M, and define a global vector field ν along ∂M by

$$\nu = \sum_\alpha \psi_\alpha \nu_\alpha.$$

Clearly ν is a smooth vector field along ∂M.
- To show that it is **outward-pointing**, let $(y^1, \cdots, y^n)$ be any smooth boundary coordinates in a neighborhood of $p \in \partial M$.

— Because each ν_α is outward-pointing, it satisfies $dy^n(\nu_\alpha) < 0$.
The y^n-component of ν at p satisfies

$$dy^n(\nu_p) = \sum_\alpha \psi_\alpha(p) dy^n(\nu_\alpha\Big|_p).$$

This sum is strictly negative, because each term is nonpositive and at least one term is negative. $\square$

Proposition 6 (The induced Orientation on a Boundary). *Let M be an oriented smooth manifold with boundary.*
Then ∂M is orientable,
and the orientation determined by any outward-pointing vector field along ∂M is independent of the choice of vector field.

Remark. The orientation on ∂M determined by any outward-pointing vactor field is called the **induced orientation** or the **Stokes orientation** on ∂M.

Proof. Let $n =\dim M$, and let Ω be an orientation form for M.
By Lemma 5, there exists a smooth outward-pointing vector field ν along ∂M.
By Proposition 2, the $(n-1)$-form $\nu \lrcorner \Omega\Big|_{\partial M}$ is an orientation form for ∂M.
Hence ∂M is orientable.
It remains to **claim: this orientation is independent of the choice of** ν.
— Indeed, let $(x^1, \cdots, x^n)$ be smooth boundary coordinates for M in a neighborhood of $p \in \partial M$.
– Replacing x^1 by $-x^1$ if necessary, we may assume that they are oriented coordinates, which implies that

$$\Omega = f\, dx^1 \wedge \cdots \wedge dx^n, \quad \text{for some strictly positive function } f.$$

Thus, using the **antiderivative** property of i_X, we have

$$(\nu \lrcorner \Omega)\Big|_{\partial M} = f \sum_{i=1}^{n} (-1)^{i-1} dx^i(\nu) dx^1\Big|_{\partial M} \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n\Big|_{\partial M}.$$

Since $x^n = 0$ along Γ, the restriction $dx^n\Big|_{\partial M} \equiv 0$. Therefore

$$(\nu \lrcorner \Omega)\Big|_{\partial M} = (-1)^{n-1} f\, dx^n(\nu) dx^1\Big|_{\partial M} \wedge \cdots \wedge dx^{n-1}\Big|_{\partial M}.$$

Since $dx^n(X) = X^n < 0$, $(\nu \lrcorner \Omega)\big|_{\partial M}$ is s positive multiple of

$$(-1)^n dx^1\Big|_{\partial M} \wedge \cdots \wedge dx^{n-1}\Big|_{\partial M}.$$

— If $\widetilde{\nu}$ is any other outward-pointing vector field,
the same computation shows that $(\widetilde{\nu} \lrcorner \Omega)\big|_{\partial M}$ is a positive multiple of $(\nu \lrcorner \Omega)\big|_{\partial M}$.
– This proves that X and $\widetilde{X}$ determine the same orientation of ∂M. □

Example. The proposition proves that $\mathbb{S}^n$ is oriented, because it is the boundary of the closed unit ball.

Example. Let us determine the induced orientation on $\mathbb{R}^n_+$ when $\mathbb{R}^n_+$ itself has the standard orientation inherited from $\mathbb{R}^n$.
— We can identify $\partial \mathbb{R}^n_+$ with $\mathbb{R}^{n-1}$ under the correspondence

$$(x^1, \cdots, x^{n-1}, 0) \leftrightarrow (x^1, \cdots, x^{n-1}).$$

Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\partial\mathbb{R}^n_+$,
the standard coordinate frame for $\mathbb{R}^{n-1}$ is positively oriented for $\mathbb{R}^n_+$ iff

$$[-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{n-1}}]$$

is the standard orientation for $\mathbb{R}^n$. This orientation satisfies

$$\begin{aligned}[-\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{n-1}}] =& -[\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{n-1}}] \\ =&(-1)^n[\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{n-1}}, \frac{\partial}{\partial x^n}].\end{aligned}$$

Thus the induced orientation for $\partial\mathbb{R}^n_+$ is equal to the standard orientation for $\mathbb{R}^{n-1}$ when n is even, but it is **opposite** to the standard orientation when n is odd.

- For many purposes, the most useful way of describing submanifolds is by means of local parametrizations.

– The next lemma gives a useful criterion for checking whether a local parametrization of a boundary is orientation-preserving.

Lemma 7. *Let M be an oriented n-manifold with boundary,*
and let $X : U \to M$ be a smooth local parametrization of ∂M, where U is a connected open subset of $\mathbb{R}^{n-1}$.
Suppose that for some $b < c \in \mathbb{R}$, X admits an extnsion to a smooth immersion

$$\widetilde{X} : (b, c] \times U \to M$$

such that $\widetilde{X}(c, x) = X(x)$.
Then X is orientation-preserving for ∂M (with the induced orientation) iff $\widetilde{X}$ is orientation-preserving for M.

Proof. Let a be an arbitrary point of U, and let

$$p = X(a) = \widetilde{X}(c, a) \in \partial M.$$

The hypothesis that $\widetilde{X}$ is an immersion means that

$$\widetilde{X}_* : (T_c\mathbb{R} \oplus T_a\mathbb{R}^{n-1}) \to T_pM$$

is injective. Since the restriction of $\widetilde{X}_*$ to $T_a\mathbb{R}^{n-1}$ is equal to

$$X_* : T_a\mathbb{R}^{n-1} \to T_p\partial M,$$

which is already injective, it follows that

$$\widetilde{X}_*(\frac{\partial}{\partial s}) \notin T_p\partial M \quad \text{where } s \text{ denotes the coordinate on } (b, c].$$

- **Claim:** $\widetilde{X}_*(\frac{\partial}{\partial s})\big|_{(c,a)}$ **is outward-pointing at** p**.**
  Indded, define a smooth curve $\gamma : [0, \varepsilon] \to M$ by
  $$\gamma(t) = \widetilde{X}(c - t, a).$$
  This curve satisfies
  $$\gamma(0) = p \qquad \gamma'(0) = -\widetilde{X}_*(\frac{\partial}{\partial s})\big|_{(c,a)}.$$
  It follows that $-\widetilde{X}_*(\frac{\partial}{\partial s})\big|_{(c,a)}$ is inwrad-pointing.
- By continuity, $\widetilde{X}_*(\frac{\partial}{\partial s})\big|_{(c,a)}$ is outward-pointing on all of $X(U)$.
- By definition of the induced orientation on ∂M,
  $\widetilde{X}$ is orientation-preserving for M
  $\Leftrightarrow (\widetilde{X}_*\frac{\partial}{\partial s}, \widetilde{X}_*\frac{\partial}{\partial x^1}, \cdots, \widetilde{X}_*\frac{\partial}{\partial x^{n-1}}$ is oriented for TM.
  $\Leftrightarrow (X_*\frac{\partial}{\partial x^1}, \cdots, X_*\frac{\partial}{\partial x^{n-1}})$ is oriented for $T\partial M$.
  $\Leftrightarrow X$ is orientation-preserving for ∂M. $\square$

Example. Spherical coordinates yield a smooth local parametrization of $\mathbb{S}^2$ as follows.
Let U be the open rectangle $(0, \pi) \times (0, 2\pi) \subset \mathbb{R}^2$,
and let $X : U \to \mathbb{R}^3$ be the following map
$$X(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi),$$
which is the restriction of the 3-dimensional spherical coordinate parametrization $\widetilde{X} : (0, 1] \times U \to \overline{\mathbb{B}^3}$ defined by
$$\widetilde{X}(\rho, \varphi, \theta) = (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi),$$
By direct computation, the Jacobian determinant of $\widetilde{X}$ is $\rho^2 \sin\varphi$, which is positive on $(0, 1) \times U$.
By Lemma 7, X is orientation-preserving.