Manifolds of Constant Curvature

- In Riemannian normal coordinates, half of the Jacobi fields are easy to write down explicitly.

Lemma 1. *Let $p \in M$, let (x^i) be normal coordinates on a nbhd $\mathcal{U}$ of p, and let γ be a radial geodesic starting at p. For any $W = W^i \partial_i \in T_pM$, the Jacobi field J along γ such that $J(0) = 0$ and $D_tJ(0) = W$ is given in normal coordinates by*

$$J(t) = tW^i \partial_i. \tag{1}$$

Proof. An easy consequence using formula for covariant derivatives in coordinates shows that J satisfies the specified initial conditions.
⊙ Thus it suffices to **claim: J is a Jacobi field**.
— If we set $V = \gamma'(0) \in T_pM$, then we know that γ is given in coordinates by

$$\gamma(t) = (tV^1, \cdots, tV^n).$$

Now consider the variation Γ given in coordinates by

$$\Gamma(s,t) = (t(V^1 + sW^1), \cdots, t(V^n + sW^n)).$$

We see that Γ is a variation through geodesics.
- Therefore its variation field $\partial_s \Gamma(0,t)$ is a Jacobi field.
- Differentiating $\Gamma(s,t)$ with restect to s shows that its variation field is $J(t)$. □

- For metrics with constant sectional curvature, we have a different kind of explicit formula for Jacobi fields—this one expresses a Jacobi field as a scalar multiple of a parallel vector field.

Lemma 2. *Suppose (M, g) is a Riemannian manifold with constant sectional curvature C, and γ is a unit speed geodesic in M. The normal Jacobi fields along γ vanishing at $t = 0$ are precisely the vector fields*

$$J(t) = u(t)E(t),$$

where E is any parallel normal vector field along γ, and $u(t)$ is given by

$$u(t) = \begin{cases} t, & C = 0; \\ R\sin\frac{t}{R}, & C = \frac{1}{R^2} > 0; \\ R\sinh\frac{t}{R}, & C = -\frac{1}{R^2} < 0. \end{cases} \tag{2}$$

Proof. Since g has constant curvature, its curvature endomorphism is given by

$$R(X,Y)Z = C(\langle Y,Z\rangle X - \langle X,Z\rangle Y).$$

Substituting this into the Jacobi equation, we find that a normal Jacobi field J satisfies

$$\begin{aligned} 0 =& D_t^2 J + C(\langle Y,Z\rangle X - \langle X,Z\rangle Y) \\ =& D^2 J + CJ, \end{aligned} \tag{3}$$

where we have used the fact that $|\gamma'| = 1$ and $\langle J, \gamma'\rangle = 0$.

— Since (3) says that the second covariant derivative of J is a multiple of J itself, it is reasonable to try to construct a solution by choosing a parallel normal vector field E along γ and setting $J(t) = u(t)E(t)$ for some solution u to be determined.

— Plugging this into (3), we find that J is a Jacobi field provided u is a solution to the differential equation

$$u'' + Cu(t) = 0.$$

It is an easy matter to sove this ODE explicitly.

— In particular, the solutions satisfying $u(0) = 0$ are constant multiple of the functions given in (2).

— The construction yields **all** the normal Jacobi fields vanishing at 0, since there is an $(n-1)$-dimensional space of them, and the space of parallel normal vector fields has the same dimension. □

- Combining the formula in the last two lemmas, we obtain our first application of Jacobi fields: **explicit expressions for constant curvature metrics in normal coordinates**.

Proposition 3. *Let (M, g) be of constant sectional curvature C.*

– *Let (x^i) be Riemannian normal coordinates on a normal nbhd $\mathcal{U}$ of p, let $|\cdot|_{\overline{g}}$ be the Euclidean norm in these coordinates, and let τ be the radial distance function.*

– *For any $g \in \mathcal{U} \setminus \{p\}$ and $V \in T_pM$, write $V = V^\top + V^\perp$, where $V^\top$ is tangent to the sphere $\{r = constant\}$ through q and $V^\perp$ is a multiple of $\frac{\partial}{\partial r}$. The metric can be written*

$$g(V,V) = \begin{cases} |V^\perp|^2_{\overline{g}} + |V^\top|^2_{\overline{g}}, & C = 0; \\ |V^\perp|^2_{\overline{g}} + \frac{R^2}{r^2}(\sin^2 \frac{r}{R})|V^\top|^2_{\overline{g}}, & C = \frac{1}{R^2} > 0; \\ |V^\perp|^2_{\overline{g}} + \frac{R^2}{r^2}(\sinh^2 \frac{r}{R})|V^\top|^2_{\overline{g}}, & C = -\frac{1}{R^2} < 0. \end{cases} \tag{4}$$

Proof. By the Gauss lemma, the decomposition $V = V^\top + V^\perp$ is orthogonal, so

$$|V|^2_g = |V^\top|^2_g + |V^\perp|^2_g.$$

Since $\frac{\partial}{\partial r}$ is a unit vector in both the g and $\overline{g}$ norms, it is immediate that

$$|V^\perp|_g = |V^\perp|_{\overline{g}}.$$

Thus we need only compute $|V^\top|_g$.

⊙ Set $X = V^\top$, and let γ denote the unit speed radial geodesic from p to q.

— By Lemma 1, X is the value of a Jacobi field J along γ that vanishes at p, namely $X = J(r)$, where $r = d(p, q)$ and

$$J(t) = \frac{t}{r} X^i \partial_i. \tag{5}$$

Because J is orthogonal to γ' at p and q, it is normal.

— Now J can also be written in the form $J(t) = u(t)E(t)$ as in Lemma 2. In this representation,

$$D_t J(0) = u'(0)E(0) = E(0),$$

since $u'(0) = 1$ in each of the cases of (2).

— Therefore, since E ia parallel and thus **of constant length**,

$$|X|^2 = |J(r)|^2 = |u(r)|^2|E(r)|^2 = |u(r)|^2|E(0)|^2 = |u(r)|^2|D_t J(0)|^2. \tag{6}$$

— Observe that

$$D_t J(0) = \frac{1}{r} X^i \partial_i \big|_p$$

by (5). Since g agrees with $\overline{g}$ at p, we have

$$|D_t J(0)| = \frac{1}{r} \Big| X^i \partial_i|_g \Big|_g = \frac{1}{r}|X|_{\overline{g}}.$$

Inserting this into (6) and using (5) for $u(r)$ completes the proof. $\square$

Proposition 4 (Local Uniqueness of Constant Curvature Metrics). *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds with constant sectional curvature C. For any points $p \in M$, $\widetilde{p} \in \widetilde{M}$, there exist nbhds $\mathcal{U}$ of p and $\widetilde{\mathcal{U}}$ of $\widetilde{p}$ and an isometry $F : \mathcal{U} \to \widetilde{\mathcal{U}}$.*

Proof. Choose $p \in M$ and $\widetilde{p} \in \widetilde{M}$, and let $\mathcal{U}$ and $\widetilde{U}$ be geodesic balls of small radius ε around p and $\tilde{p}$, respectively.

— Riemannian normal coordinated give maps

$$\varphi : \mathcal{U} \to B_\varepsilon(0) \subset \mathbb{R}^n \quad \text{and} \quad \widetilde{\varphi} : \widetilde{\mathcal{U}} \to B_\varepsilon(0) \subset \mathbb{R}^n,$$

under which both metrics are given by (4).

— Therefore $\widetilde{\varphi}^{-1} \circ \varphi$ is the required local isometry. $\square$