Manifolds of Constant Curvature

- In Riemannian normal coordinates, half of the Jacobi fields are easy to write down explicitly.

Lemma 1. *Let $p \in M$, let (x^i) be normal coordinates on a nbhd $\mathcal{U}$ of p, and let γ be a radial geodesic starting at p. For any $W = W^i\partial_i \in T_pM$, the Jacobi field J along γ such that $J(0) = 0$ and $D_tJ(0) = W$ is given in normal coordinates by*

$$J(t) = tW^i\partial_i. \tag{1}$$

- For metrics with constant sectional curvature, we have a different kind of explicit formula for Jacobi fields—this one expresses a Jacobi field as a scalar multiple of a parallel vector field.

Lemma 2. *Suppose (M, g) is a Riemannian manifold with constant sectional curvature C, and γ is a unit speed geodesic in M. The normal Jacobi fields along γ vanishing at $t = 0$ are precisely the vector fields*

$$J(t) = u(t)E(t),$$

where E is any parallel normal vector field along γ, and $u(t)$ is given by

$$u(t) = \begin{cases} t, & C = 0; \\ R\sin\frac{t}{R}, & C = \frac{1}{R^2} > 0; \\ R\sinh\frac{t}{R}, & C = -\frac{1}{R^2} < 0. \end{cases} \tag{2}$$

- Combining the formula in the last two lemmas, we obtain our first application of Jacobi fields: **explicit expressions for constant curvature metrics in normal coordinates**.

Proposition 3. *Let (M, g) be of constant sectional curvature C.*
– Let (x^i) be Riemannian normal coordinates on a normal nbhd $\mathcal{U}$ of p, let $|\cdot|_{\overline{g}}$ be the Euclidean norm in these coordinates, and let τ be the radial distance function.
– For any $g \in \mathcal{U} \setminus \{p\}$ and $V \in T_pM$, write $V = V^\top + V^\perp$, where $V^\top$ is tangent to the sphere $\{r = constant\}$ through q and $V^\perp$ is a multiple of $\frac{\partial}{\partial r}$. The metric can be written

$$g(V, V) = \begin{cases} |V^\perp|^2_{\overline{g}} + |V^\top|^2_{\overline{g}}, & C = 0; \\ |V^\perp|^2_{\overline{g}} + \frac{R^2}{r^2}(\sin^2\frac{r}{R})|V^\top|^2_{\overline{g}}, & C = \frac{1}{R^2} > 0; \\ |V^\perp|^2_{\overline{g}} + \frac{R^2}{r^2}(\sinh^2\frac{r}{R})|V^\top|^2_{\overline{g}}, & C = -\frac{1}{R^2} < 0. \end{cases} \tag{4}$$

Proposition 4 (Local Uniqueness of Constant Curvature Metrics). *Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be Riemannian manifolds with constant sectional curvature C. For any points $p \in M$, $\widetilde{p} \in \widetilde{M}$, there exist nbhds $\mathcal{U}$ of p and $\widetilde{\mathcal{U}}$ of $\widetilde{p}$ and an isometry $F : \mathcal{U} \to \widetilde{\mathcal{U}}$.*

Proof. Choose $p \in M$ and $\widetilde{p} \in \widetilde{M}$, and let $\mathcal{U}$ and $\widetilde{U}$ be geodesic balls of small radius ε around p and $\tilde{p}$, respectively. Riemannian normal coordinated give maps

$$\varphi : \mathcal{U} \to B_\varepsilon(0) \subset \mathbb{R}^n \quad \text{and} \quad \widetilde{\varphi} : \widetilde{\mathcal{U}} \to B_\varepsilon(0) \subset \mathbb{R}^n,$$

under which both metrics are given by (4). Therefore $\widetilde{\varphi}^{-1} \circ \varphi$ is the required local isometry. □

Theorem 5. *Suppose $\widetilde{M}$ and M are connected Riemannian manifolds, with $\widetilde{M}$ complete, and $\pi : \widetilde{M} \to M$ is a local isometry. Then M is complete and π is a covering map.*

Theorem 6 (Uniqueness of Constant Curvature Metrics). *Let M be a complete, simply-connected Riemannian n-manifold with constant sectional curvature C. Then M is isometric to one of the model spaces $\mathbb{R}^n$, $\mathbb{S}^n_R$ or $\mathbb{H}^n_R$.*

Proof. Handle the cases of positive and nonpositive sectional curvature seperately.

(I) First suppose $C \leq 0$. Choosing any point $Q \in \mathbb{H}^n_R$ and any linear isometry $\varphi : T_Q\mathbb{H}^n_R \to T_pM$, the map $\Phi : \mathbb{H}^n_R \to M$ given by

$$\Phi : \exp_p \circ \varphi \circ \exp_Q^{-1}$$

is a globally defined local isometry.

— By Theorem 5, $\Phi : \mathbb{H}^n_R \to M$ is covering map for any $p \in M$.

— Since M is simply-connected, it is a diffeomorphism.

(II) Secondly suppose $C = \frac{1}{R^2} > 0$.

⊙ Let $\{N, -N\}$ be the north and south poles in $\mathbb{S}^n_R$,
and observe that $\exp_N$ is a diffeomorphism from $B_{\pi R}(0) \subset T_N\mathbb{S}^n_R$ to $\mathbb{S}^n_R \setminus \{-N\}$.

— On the other hand, choosing any point $p \in M$, **the conjugate point comparison theorem** shows that p has no conjugate point closer than πR,
so $\exp_p$ is at least a local diffeomorphism on $B_{\pi R}(0) \subset T_pM$.

— If we choose any linear isometry $\varphi : T_N\mathbb{S}^n_R \to T_pM$, then $(\exp_p \circ\varphi)^* g$ and $\exp_N^* \mathring{g}_R$ are both metrics of constant curvature $\frac{1}{R^2}$ on $B_R(0) \subset T_N\mathbb{S}^n_R$,
and Euclidean coordinates on $T_N\mathbb{S}^n_R$ are normal coordinates for both
(since the radial line segments are geodesics).

— Therefore, Proposition 4 shows that they are equal, so the map $\Phi : \mathbb{S}^n_R \setminus \{-N\} \to M$ given by

$$\Phi : \exp_p \circ \varphi \circ \exp_N^{-1}$$

is a local isometry.

⊙ Now choose any point $Q \in \mathbb{S}^n_R$ other than N or $-N$, and let $q = \Phi(Q) \in M$.

— Using the isometry

$$\widetilde{\varphi} = \Phi_* : T_Q\mathbb{S}^n_R \to T_qM,$$

we can construct a similar map

$$\widetilde{\Phi} = \exp_q \circ \widetilde{\varphi} \circ \exp_Q^{-1} : \mathbb{S}_R^n \setminus \{-Q\} \to M,$$

and the same argument shows that $\widetilde{\Phi}$ is a local isometry.

— Because $\Phi(Q) = \widetilde{\Phi}(Q)$ and $\Phi_* = \widetilde{\Phi}_*$ at Q by construction, Φ and $\widetilde{\Phi}$ must agree where they overlap.

— Putting them together, therefore, we obtain a globally defined local isometry $F : \mathbb{S}_R^n \to M$. $\square$

- Theorem 6 is a special case of a rather more complicated result, **the Cartan-Ambrose-Hicks theorem**, which says roughly that **two simply-connected manifolds, all of whose sectional curvatures at corresponding points are equal to each other, must be isometric**.

— The main idea of the proof is very similar to what we have done here; the trick is in making precise sense of the notion of "corresponding points", and of what it means for nonconstant sectional curvatures to be equal at different points of different manifolds.

Corollary 7 (Classification of Constant Curvature Metrics). *Suppose M is a complete, connected Riemannian manifold with constant sectional curvature. Then M is isometric to $\widetilde{M}/\Gamma$, where $\widetilde{M}$ is one of the constant curvature medel spaces $\mathbb{R}^n$, $\mathbb{S}_R^n$ or $\mathbb{H}_R^n$, and Γ is a discrete subgroup of $\mathcal{T}(\widetilde{M})$,* **isometric to** *$\pi_1(M)$, and acting freely and properly discontinuous on $\widetilde{M}$.*

Proof. If $\pi : \widetilde{M} \to M$ is the universal covering of M with the lifted metric $\tilde{g} = \pi^* g$, the preceeding theorem shows that $(\widetilde{M}, \tilde{g})$ is isometric to one of the model spaces.

— From covering space theory it follows that **the group Γ of covering (deck) transformations** is isometric to $\pi_1(M)$
and acts freely and properly discontinuously on $\widetilde{M}$,
and M is diffeomorphic to the quotient $\widetilde{M}/\Gamma$.

— Moreover, if φ is any covering transformation, $\pi \circ \varphi = \pi$, and so

$$\varphi^* \widetilde{g} = \varphi^* \pi^* g = \pi^* g = \tilde{g}$$

so Γ acts by isometries.

— **Claim: Γ is discrete in $\mathcal{T}(\widetilde{M})$.**

– Suppose $\{\varphi_i\} \subset \Gamma$ is an infinite set with an accumulation point in $\mathcal{T}(M)$.

– Since the action of Γ is fixed-poinf free, for any point $\tilde{p} \in \widetilde{M}$ the set $\{\varphi_i(\widetilde{p})\}$ is infinite, and by **continuity** of the action, it has an accumulation point in $\widetilde{M}$.

– But this is impossible, since the points $\{\varphi_i(\widetilde{p})\}$ all project to the same point in M, and so form a discrete set. $\square$

Definition. *A complete, connected Riemannian manifold with constant sectional curvature is called a* **space form**.

- This result essentially reduces the classification of space forms to group theory.