Densities

- Although differential forms are natural objects to integrate on manifolds, and are essential for use in Stoke's theorem, they have the disadvantage of requiring oriented manifolds in order for their integrals to be defined.
- There is a way to define integration on nonorientable manifolds as well, which we describe below.

- The reason an orientation is needed for integrals of differential forms to make sense has to do with transformation law under change of coordinates.

— The transformation law for an n-form on an n-manifold under a change of coordinates involves the jacobian determinant of the transition map,
while the transformation law for integrals involves the absolute value of the determinant.

— We will define below objects whose transformation law for integrals involves the absolute value of the determinant.

- We begin, as always, in the linear-algebraic setting.

Definition. *Let V be an n-dimensional vector space. A* **density** *of V is a function*

$$\mu : \underbrace{V \times \cdots \times V}_{n \text{ copies}} \to \mathbb{R}$$

satisfying the following condition: If $T : V \to V$ is any linear map, then

$$\mu(X_1, \cdots, X_n) = |\det T|\, \mu(X_1, \cdots, X_n).$$

- Observe that a density is **not** a tensor, because it is not linear over $\mathbb{R}$ in any of its arguments.

Let $\Omega(V)$ denote the set of all densities on V.

Proposition 1 (Properties of Densities). *Let V be a vector space of $\dim V = n \geq 1$.*
(a) $\Omega(V)$ is a vector space under the obvious vector operations:

$$(c_2\mu_1 + c_2\mu_2)(X_1, \cdots, X_n) = c_1\mu_1(X_1, \cdots, X_n) + c_2\mu_2(X_1, \cdots, X_n).$$

(b) If μ_1, $\mu_2 \in \Omega(V)$ and $\mu_1(E_1, \cdots, E_n) = \mu_2(E_1, \cdots, E_n)$ for some basis (E_i) of V, then $\mu_1 = \mu_2$.
(c) If $\omega \in \Lambda^n(V)$, the map $|\omega| : V \times \cdots \times V \to \mathbb{R}$ defined by

$$|\omega|(X_1, \cdots, X_n) = |\omega(X_1, \cdots, X_n)|$$

is a density
(d) $\Omega(V)$ is a 1-dimensional spanned by $|\omega|$ for any nonzero $\omega \in \Lambda^n(V)$.

Proof. (b) Suppose μ_1 and μ_2 give the sams value when applied to $(E_1, \cdots, E_n)$.

If $X_1, \cdots, X_n$ are arbitrary vectors in V, let $T : V \to V$ be the unique linear map that takes E_i to X_i, for $i = 1, \cdots, n$. It follows that

$$\begin{aligned}\mu_1(X_1, \cdots, X_n) =&\mu_1(TE_1, \cdots, TE_n)\\ =&|\det T|\mu_1(E_1, \cdots, E_n)\\ =&|\det T|\mu_2(E_1, \cdots, E_n)\\ =&\mu_2(TE_1, \cdots, TE_n)\\ =&\mu_2(X_1, \cdots, X_n).\end{aligned}$$

(c) We have

$$\begin{aligned}|\omega|(TX_1, \cdots, TX_n) =&|\omega(TX_1, \cdots, TX_n)|\\ =&|(\det T)\omega(X_1, \cdots, X_n)|\\ =&|\det T||\omega|(X_1, \cdots, X_n).\end{aligned}$$

(d) Suppose ω is any nonzero element of $\Lambda^n(V)$.
It suffices to **claim:** $\forall \mu \in \Omega(V)$**,** $\exists c \in \mathbb{R}$ **such that** $\mu = c|\omega|$**.**

– Let (E_i) be a basis for V, and define a, $b \in \mathbb{R}$ by

$$\begin{aligned}a =&|\omega|(E_1, \cdots, E_n) = |\omega(E_1, \cdots, E_n)|,\\ b =&\mu(E_1, \cdots, E_n).\end{aligned}$$

Since $\omega \neq 0$, we have $a \neq 0$. Thus

$$\mu(E_1, \cdots, E_n) = (b/a)|\omega|(E_1, \cdots, E_n).$$

Hence $\mu = (b/a)|\omega|$ by part (b). $\square$

Definition. *(1) A* **positive density** *on V is a density μ with $\mu(X_1, \cdots, X_n) > 0$ for all linearly independent $(X_1, \cdots, X_n)$.*
(2) A **negative density** *is defined similarly.*

- Every density on V is either positive, negative or zero.
- The set of positive densities is a convex subset of $\Omega(V)$, namely, a half-line.

- Now let M be a smooth manifold.

Definition. *The set*

$$\Omega M = \prod_{p \in M} \Omega(T_pM)$$

is called the **density bundle** *of M.*
Let $\pi : \Omega M \to M$ be the projection map taking each element of $\Omega(T_pM)$ to p.

Lemma 2. *If M is a smooth manifold, its density bundle is a smooth line bundle over M.*

Proof. We will construct local trivilizations.

— Let $(U, (x^i))$ be any smooth coordinate chart on M, and let

$$\omega = dx^1 \wedge \cdots \wedge dx^n.$$

Proposition 1 shows that $|\omega_p|$ is a basis for $\Omega(T_pM)$ at each point $p \in U$. Therefore, the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}$ given by

$$\Phi(c|\omega_p|) = (p, c)$$

is a bijecton.

- Suppose $(\widetilde{U}, (\widetilde{x}^j))$ is another smooth chart with $U \cap \widetilde{U} \neq \emptyset$. Let

$$\widetilde{\omega} = d\widetilde{x}^1 \wedge \cdots \wedge d\widetilde{x}^n,$$

and define $\widetilde{\Phi} : \pi^{-1}(\widetilde{U}) \to \widetilde{U} \times \mathbb{R}$ corresponding;y:

$$\Phi(c|\widetilde{\omega}_p|) = (p, c).$$

Then

$$\begin{aligned} \Phi \circ \widetilde{\Phi}^{-1}(p, c) =& \Phi(c|\widetilde{\omega}_p|) = \Phi(c\Big| \det(\frac{\partial \widetilde{x}^j}{\partial x^i})\Big||\omega_p|) \\ =& (p, c\Big| \det\left(\frac{\partial \widetilde{x}^j}{\partial x^i}\right)\Big|). \end{aligned}$$

Thus the transition functions are equal to $\det\Big(\partial \widetilde{x}^j / \partial x^i\Big)$. □

Definition. *A section ΩM is called a* **density** *on M.*

— *A density on M is said to be positive or negative if its value at each point has that property.*

- Any nonvanishing n-form ω determines a positive density $|\omega|$, defined by $|\omega|_p = |\omega_p|$ for each $p \in M$.
  If ω is a nonvanishing n-form on an open set $U \subset M$, then any density μ on U can be written $\mu = f|\omega|$ for some real-valued function f.
- One important fact about densities is that every manifold admits a **global** smooth positive density, without any orientability assumptions.

Lemma 3. *If M is a smooth manifold, there exists a smooth positive density on M.*

Proof. Because the set pf positive elements of ΩM is an open subset whose intersection with each fiber is **convex**, the usual partition of unity argument allows us to piece togeter local densities to obtain a global smooth positive density. □

Remark. This lemma works because **positivity of a density** is a well-defined property, independent of any choices of coordinates or orientations.

— There is no corresponding existence result for orientation forms because without a choice of orientations, there is no way to decide whether n-forms are positive.

- Under smooth maps, densities pull back in the same way as differential forms.

Definition. *If $F : M \to N$ is a smooth map between n-manifolds and μ is a density on N, we define a density $F^*\mu$ on M by*

$$(F^*\mu)_p(X_1\Big|_p, \cdots, X_n\Big|_p) = \mu_{F(p)}(F_*X_1\Big|_p, \cdots, F_*X_n\Big|_p).$$

Lemma 4. *Let $G : P \to N$ and $F : M \to N$ be smooth maps between n-manifolds, and let μ be a density on N.*
(a) For any $f \in C^\infty(N)$, $F^*(f\mu) = (f \circ F)F^*\mu$.
(b) If ω is an n-form on N, then $F^*|\omega| = |F^*\omega|$.
(c) If μ is smooth, then $F^*\mu$ is a smooth density on M.
(d) $(F \circ G)^*\mu = G^*(F^*\mu)$.

- The next result shows how to compute the pullback of a density in coordinates.

Proposition 5. *Suppose $F : M \to N$ is a smooth map between n-manifolds. If (x^i) and (y^j) are smooth coordinates on open sets $U \subset M$ and $V \subset N$, respectively, and u is a smooth real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:*

$$F^*(u|dy^1 \wedge \cdots \wedge dy^n|) = (u \circ F)|\det DF||dx^1 \wedge \cdots \wedge dx^n|, \tag{1}$$

where DF represents the matrix of partial derivatives of F in these coordinates.

Proof. We have

$$\begin{aligned} F^*(u|dy^1 \wedge \cdots \wedge dy^n|) =& (u \circ F)F^*(|dy^1 \wedge \cdots \wedge dy^n|) \\ =& (u \circ F)|F^*(dy^1 \wedge \cdots \wedge dy^n)|, \quad \text{by Lemma 4(b)}, \\ =& (u \circ F)|(\det DF)dx^1 \wedge \cdots \wedge dx^n| \\ =& (u \circ F)|\det DF||dx^1 \wedge \cdots \wedge dx^n|. \qquad \square \end{aligned}$$

Integration of Densities

- If $D \subset \mathbb{R}^n$ is a compact domain of integration and μ is a density on D, we can write
$$\mu = f\,|dx^1 \wedge \cdots \wedge dx^n|$$
for some uniquely determined continuous function $f : D \to \mathbb{R}$.

Definition. *Define the* **integral of** μ **over** D *by*
$$\int_D \mu = \int_D f\,dV,$$
or more suggestively,
$$\int_D f\,|dx^1 \wedge \cdots \wedge dx^n| = \int_D f\,dx^1 \cdots dx^n.$$

- *Similarly, if* U *is an open subset of* $\mathbb{R}^n$ *and* μ *is compactly supported in* U*, we define*
$$\int_U \mu = \int_D \mu,$$
where $D \subset U$ *is any compact domain of integration containing the support of* μ*.*

- The key fact is that this is diffeomorphism-invariant.

Proposition 6. *(1) If* D *and* E *are compact domains of integration in* $\mathbb{R}^n$*, and* $G : D \to E$ *is a smooth map that restricts to a diffeomorphism from* $\operatorname{Int} D$ *to* $\operatorname{Int} E$*, then*
$$\int_E \mu = \int_D G^*\mu$$
for any density μ *on* E*.*
(2) Similarly, if $U, V \subset \mathbb{R}^n$ *are open sets and* $G : U \to V$ *is a diffeomorphism, then*
$$\int_V \mu = \int_U G^*\mu$$
for any compactly supported density μ *on* V*.*

Proof. Use (1) in Proposition 5. □

- Now let M be a smooth n-manifold. If μ is a density on M whose support is contained in the domain of a single smooth chart (U, φ), the **integral of** μ **over** M is defined as
$$\int_M \mu = \int_{\varphi(U)} (\varphi)^{-1}\mu.$$

— This is extended to arbitrary densities μ by setting
$$\int_M \mu = \sum_i \int_M \psi_i \mu,$$
where $\{\psi_i\}$ is a smooth partition of unity subordinate to an open cover of M by smooth charts.

— This is independent of the choices of coordinates or partition of unity.

Proposition 7 (Properties of Integrals of Densities). *Suppose M and N are smooth manifolds with or without boundaries, and μ, ν are compactly supported densities on M.*
(a) **Linearity:** *If $a, b \in \mathbb{R}$, then $\int_M a\mu + b\eta = a\int_M \mu + b\int_M \eta$.*
(b) **Positivity:** *If μ is a positive density, then $\int_M \mu > 0$.*
(c) **Diffeomorphism Invariance:** *If $F : N \to M$ is a diffeomorphism, then*

$$\int_M \mu = \int_N F^* \mu.$$

The Riemannian Density

Lemma 8 (The Riemannian Density). *Let (M, g) be a Riemannian manifold with or without boundary. There is a unique smooth positive density μ on M, called the* **Riemannian density**, *with the property that*

$$\mu(E_1, \cdots, E_n) = 1 \tag{14}$$

for any local orthonormal frame (E_i).

Proof. **(i) Uniqueness** is obvious, because any two densities that agree on the elements of a basis must be equal.

(ii) Existence: Given any point $p \in M$, let U be a connected smooth coordinate neighborhood of p.
— Since U is diffeomorphic to an open subset of Euclidean space, it is orientable.
— Any choice of orientation of U uniquely determines a Riemannian volume form dv_g with the property that

$$dv_g(E_1, \cdots, E_n) = 1$$

for any oriented orthonormal frame.
— If we put $\mu = |dv_g|$, it follows easily that μ is a smooth positive density on U satisfying (14).
— If U and V are two overlapping smooth coordinate neighborhoods, the two definitions of μ agree where they overlap by **uniqueness**, so this defines μ globally. □

Proposition. *Let (M, g) be a Riemannian manifold with or without boundary and let dv_g be its Riemannian volume form.*
(a) The Riemannian density of M ie equal to $|dv_g|$.
(b) For every compactly supported continuous function $f : M \to \mathbb{R}$,

$$\int_M f\,|dv_g| = \int_M f\,dv_g.$$

- Because of part (b), it is customary to denote the Riemannian density simply by dv_g, and to specify when necessary whether the notation refers to a density or a form.