Proposition 1 (Exterior Derivative of a 1-Form). *For any smooth* 1*-form* ω *and smooth vector fields* X *and* Y*,*

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \tag{1}$$

Proof. Since any smooth 1-form can be expressed locally as a sum of terms of the form udv for smooth functions u and v, it suffices to consider that case.
Suppose $\omega = udv$, and X, Y are smooth vector fields. Then the left-hand side of (1) is

$$\begin{aligned} d(udv)(X,Y) =& du \wedge dv(X,Y) = du(X)dv(Y) - dv(X)du(Y) \\ =& XuYv - XvYu. \end{aligned}$$

The right-hand side is

$$\begin{aligned} &X(udv(Y)) - Y(udv(X)) - udv([X,Y]) \\ &\quad = X(uYv) - Y(uXv) - u[X,Y]v \\ &\quad = (XuYv + uXYv) - (YuXv + uYXv) - u(XYv - YXv) \\ &\quad = XuYv - XvYu. \qquad \square \end{aligned}$$

- (1) shows that **the exterior derivative is in a certain sense dual to the Lie bracket**.

— In particular, it shows that if we know all the Lie brackets of basis vector fields in a smooth local frame, we can compute the exterior derivatives of the dual covector fields, and vice versa.

Proposition 2. *Let* M *be a smooth* n*-manifold, let* (E_i) *be a smooth local frame for* M*, and let* (ε^i) *be the dual coframe. Let* c^i_{jk}*,* $i = 1, \cdots, n$*, be the component functions of the Lie bracket* $[E_j, E_k]$ *in the frame:*

$$[E_j, E_k] = c^i_{jk} E_i.$$

Then the exterior derivative of each 1*-form* ε^i *is given by*

$$d\varepsilon^i = -c^i_{jk} \varepsilon^j \wedge \varepsilon^k.$$

Proposition 3 (Invariant Formula for Exterior Derivatives).
Let M *be a smooth* n*-manifold and* $\omega \in \mathcal{A}^k(M)$*. For any smooth vector fields* $X_1, \cdots, X_{k+1}$ *on* M*,*

$$\begin{aligned} d\omega(X_1, \cdots, X_{k+1}) = &\sum_{1 \le i \le k+1} (-1)^{i-1} X_i\big(\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})\big) \\ + \sum_{1 \le i \le j \le k+1} &(-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \widehat{X}_j, \cdots, X_{k+1}). \end{aligned} \tag{2}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Proof. Denote the two sums on the right-hand side of (2) by $I(X_1, \cdots, X_{k+1})$ and $II(X_1, \cdots, X_{k+1})$, and the entire right-hand side by $E\omega(X_1, \cdots, X_{k+1})$.
— Note that $E\omega$ is multilinear over $\mathbb{R}$.

- First **claim:** $E\omega$ **is multilinear over** $C^\infty(M)$, i.e., for $1 \le p \le k+1$ and $f \in C^\infty(M)$,

$$E\omega(X_1, \cdots, fX_p, \cdots, X_{k+1}) = (-1)^{p-1} f E\omega(X_1, \cdots, X_p, \cdots, X_{k+1}).$$

(i) **In the expansion of** $I(X_1, \cdots, fX_p, \cdots, X_{k+1})$,

$$\begin{aligned} E(X_1, \cdots, fX_p, \cdots, X_{k+1}) =& fX_p\big(\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})\big) \\ &+ \sum_{i \neq p} (-1)^{i-1} X_i\big(f\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})\big) \end{aligned}$$

The second term on the right-hand side expand as follows:

$$\begin{aligned} &\sum_{i \neq p} (-1)^{i-1} X_i\big(f\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})\big) \\ &\quad = \sum_{i \neq p} (-1)^{i-1} fX_i\big(\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})\big) \\ &\quad\quad + \sum_{i \neq p} (-1)^{i-1} (X_i f)\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1})\Big). \end{aligned}$$

$$\begin{aligned} \therefore I(X_1, \cdots, fX_p, \cdots, X_{k+1}) &= fI(X_1, \cdots, X_p, \cdots, X_{k+1}) \\ &+ \sum_{i \neq p} (-1)^{i-1} (X_i f)\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{k+1}). \end{aligned} \tag{3}$$

(ii) **Consider next the expansion** II.
– clearly, f factors out of all the terms in which $i \neq p$ and $j \neq p$.
– To expand the other terms, we observe that

$$\begin{aligned} [fX_p, X_j] =& f[X_p, X_j] - (X_j f)X_p, \\ [X_i, fX_p] =& f[X_i, X_p] + (X_i f)X_p. \end{aligned}$$

Inserting these formulas into the $i = p$ and $j = p$ terms, we obtain

$$\begin{aligned} II(X_1, \cdots, fX_p, \cdots, X_{k+1}) &= f\, II(X_1, \cdots, X_p, \cdots, X_{k+1}) \\ &+ \sum_{p<j} (-1)^{p+j+1} (X_j f)\omega(X_p, X_1, \cdots, \widehat{X}_p, \cdots, \widehat{X}_j, \cdots, X_{k+1}) \\ &+ \sum_{i<p} (-1)^{i+p} (X_i f)\omega(X_p, X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_p, \cdots, X_{k+1}) \end{aligned}$$

Rearranging the arguments in these two sums so as to put X_p into its original position, we see that they exactly cancel the sum in (3).

This completes the proof that $E\omega$ is multilinear over $C^\infty(M)$.

- By multinearity, to verify that $E\omega = d\omega$, it suffices to show that both sides give the same result when applied to any sequence of basis vectors in an arbitrary local frame.

— The computations are greatly simplified by working in a coordinate frame, for which all the Lie brackets vanish.

— Thus let $(U, (x^i))$ be an arbitrary smooth chart on M.

– Because both $d\omega$ and $E\omega$ depend linearly on ω, we may assume that

$$\omega = f dx^I \quad \text{for some function } f \text{ and increasing multi-index } I = (i_1, \cdots, i_k),$$

$$\therefore d\omega = df \wedge dx^I = \sum_\ell \frac{\partial f}{\partial x^\ell} dx^\ell \wedge dx^I.$$

– If $J = (j_1, \cdots, j_{k+1})$ is any multi-index of length $k+1$, it follows that

$$d\omega\left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k+1}}\right) = \sum_\ell \frac{\partial f}{\partial x^\ell} \delta^{\ell I}_J.$$

The only term in this sum that can possibly be nonzero are those for which ℓ is equal to one of the indices in J, say $\ell = j_p$.
In this case, $\delta^{\ell I}_J = (-1)^{p-1} \delta^I_{\widehat{J}_p}$, where $\widehat{J}_p = (j_1, \cdots, \widehat{j}_p, \cdots, j_{k+1})$, so

$$d\omega\left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k+1}}\right) = \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial f}{\partial x^{j_p}} \delta^I_{\widehat{J}_p}. \tag{4}$$

— On the other hand, because all the Lie brackets are zero, we have

$$\begin{aligned} &E\omega\left(\frac{\partial}{\partial x^{j_1}}, \cdots, \frac{\partial}{\partial x^{j_k+1}}\right) \\ &= \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial}{\partial x^{j_p}} \left(f dx^I \left(\frac{\partial}{\partial x^{j_1}}, \cdots, \widehat{\frac{\partial}{\partial x^{j_p}}}, \cdots, \frac{\partial}{\partial x^{j_{k+1}}} \right)\right) \\ &= \sum_{1 \le p \le k+1} (-1)^{p-1} \frac{\partial f}{\partial x^{j_p}} \delta^I_{\widehat{J}_p}, \end{aligned}$$

which agrees with (4). $\square$

- Let X be a smooth vector field on smooth mfd M, and let θ be its flow.

⊙ For any $p \in M$, it is sufficiently close to zero, θ_t is a diffeomorphism from a nbhd of p to a nbhd of $\theta_t(p)$, so θ_t^* pulls back tensors at $\theta_t(p)$ to ones at p.

Definition. *Given a smooth covariant tensor field τ on M, we define the* **Lie derivative** *of τ with respect to X, denoted by $\mathcal{L}_X\tau$, by*

$$(\mathcal{L}_X\tau)_p = \frac{d}{dt}\Big|_{t=0}(\theta_t^*\tau)_p = \lim_{t\to 0}\frac{\theta_t^*(\tau_{\theta_t(p)} - \tau_p)}{t}, \tag{5}$$

provided the derivative exists.

- Because the expression being differentiate lies in $T^k(T_pM)$ for all t, $(\mathcal{L}_X\tau)_p$ makes sense as an element of $T^k(T_pM)$

Lemma 4. *If X is a smooth vector field and τ is a smooth covaiant tensor field on a smooth manifold M, then the derivatives in (5) exists $\forall p \in M$, and the assignment $p \to (\mathcal{L}_X\tau)_p$ defines a smooth tensor field on M.*

Proposition 5. *Suppose X, Y are smooth vector fields; f is a smooth real-valued function (regarded as 0-tensor field); σ, τ are smooth covariant tensor fields; and ω, η are smooth fifferential forms.*

(a) $\mathcal{L}_X f = Xf$.

(b) $\mathcal{L}_X(f\sigma) = (\mathcal{L}_X f)\sigma + f\mathcal{L}_X\sigma$.

(c) $\mathcal{L}_X(\sigma\otimes\tau) = (\mathcal{L}_\sigma)\otimes\tau + \sigma\otimes\mathcal{L}_X\tau$.

(d) $\mathcal{L}_X(\omega\otimes\eta) = (\mathcal{L}_X\omega)\otimes\eta + \omega\otimes\mathcal{L}_X\eta$.

(e) If $Y_1,\cdots,Y_k$ are smooth vector fields and σ is a smooth k-tensor field, then

$$\begin{aligned}\mathcal{L}_X(\sigma(Y_1,\cdots,Y_k)) =& (\mathcal{L}_X\sigma)(Y_1,\cdots,Y_k)\\ &+\cdots+\sigma(Y_1,\cdots,\mathcal{L}_XY_k).\end{aligned} \tag{6}$$

Proof. (a) The first assertion is just a reinterpretation of the definition in the case of a 0-tensor field: Because $\theta_t^* f = f\circ\theta_t$, the definition implies

$$\mathcal{L}_X f(p) = \frac{d}{dt}\Big|_{t=0} f(\theta_t(p)) = Xf(p).$$

(c) We have

$$\begin{aligned}(\mathcal{L}_X(\sigma\otimes\tau))_p =& \lim_{t\to 0}\frac{\theta_t^*((\sigma\otimes\tau)_{\theta_t(p)}) - (\sigma\otimes\tau)_p}{t}\\
=& \lim_{t\to 0}\frac{\theta_t^*(\sigma_{\theta_t(p)})\otimes\theta_t^*(\tau_{\theta_t(p)}) - \sigma_p\otimes\tau_p}{t}\\
=& \lim_{t\to 0}\frac{\theta_t^*(\sigma_{\theta_t(p)})\otimes\theta_t^*(\tau_{\theta_t(p)}) - \theta_t^*(\sigma_{\theta_t(p)})\otimes\tau_p}{t}\\
&+\lim_{t\to 0}\frac{\theta_t^*(\sigma_{\theta_t(p))}\otimes\tau_p - \sigma_p\otimes\tau_p}{t}\\
=& \lim_{t\to 0}\theta_t^*(\sigma_{\theta_(p)})\otimes\frac{\theta_t^*(\tau_{\theta_t(p)}) - \tau_p}{t} + \lim_{t\to 0}\frac{\theta_t^*(\sigma_{\theta_t(p)}) - \sigma_p}{t}\otimes\tau_p\\
=&\sigma_p\otimes(\mathcal{L}_X\tau)_p + (\mathcal{L}_X\sigma)_p\otimes\tau_p. \qquad \square\end{aligned}$$

Corollary 6. *If X is a smooth vector field and σ is a smooth covariant k-tensor field, then for any smooth vector fields $Y_1, \cdots, Y_k$,*

$$(\mathcal{L}_X\sigma)(Y_1, \cdots, Y_k)) = X(\sigma(Y_1, \cdots, Y_k)) - \sigma([X, Y_1], Y_2, \cdots, Y_k) - \cdots - \sigma(Y_1, \cdots, Y_{k-1}, [X, Y_k]). \tag{7}$$

Corollary 7. *If $f \in C^\infty(M)$, then $\mathcal{L}_X(df) = d(\mathcal{L}_X f)$.*

Proof. Using (7), we compute

$$\begin{aligned}(\mathcal{L}_X df)(Y) =& X(df(Y)) - df[X, Y] = XYf - [X, Y]f \\ =& XYf - (XYf - YXf)f = YXf \\ =& d(Xf)(Y) = d(\mathcal{L}_X f)(Y). \quad \square\end{aligned}$$

Cartan formula I. *$\mathcal{L}_X(Y \lrcorner\, \omega) = (\mathcal{L}_X Y) \lrcorner\, \omega + Y \lrcorner\, \mathcal{L}_X\omega$; i.e.*

$$\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]}.$$

Proof. It is obvious for 0-form $\omega = f$. Let ω be an arbitrary k-form with $k > 0$. Then, for any $X_1, \cdots, X_{k-1} \in \Gamma(TM)$,

$$\begin{aligned}(\mathcal{L}_X i_Y\omega)(X_1, \cdots, X_{k-1}) =& X((i_Y\omega)(X_1, \cdots, X_{k-1})) \\ & - \sum_{i=1}^{k-1}(i_Y\omega)(X_1, \cdots, [X, X_i], \cdots, X_{k-1}) \\ =& X(\omega(Y, X_1, \cdots, X_{k-1})) \\ & - \sum_{i=1}^{k-1}\omega(Y, X_1, \cdots, [X, X_i], \cdots, X_{k-1}).\end{aligned} \tag{9}$$

On the other hand,

$$\begin{aligned}(i_Y\mathcal{L}_X\omega)(X_1, \cdots, X_{k-1}) =& \mathcal{L}_X\omega(Y, X_1, \cdots, X_{k-1}) \\ =& X(\omega(Y, X_1, \cdots, X_{k-1})) - \omega([X, Y], X_1, \cdots, X_{k-1}) \\ & - \sum_{i=1}^{k-1}\omega(Y, X_1, \cdots, [X, X_i], \cdots, X_{k-1}).\end{aligned} \tag{10}$$

Subtracting (10) from (9), we have

$$\mathcal{L}_X i_Y\omega - i_Y\mathcal{L}_X\omega = \omega([X, Y], X_1, \cdots, X_{k-1}) = i_{[X,Y]}\omega. \quad \square$$

Cartan's Formula II. *For any smooth vector field X and any smooth differential form ω,*

$$\mathcal{L}_X\omega = X \lrcorner (d\omega) + d(X \lrcorner \omega); \tag{11}$$

that is,

$$\mathcal{L}_X = i_X d + d i_X.$$

Proof. We will prove that (11) holds for smooth k-forms by **induction** on k.

(i) We begin with a smooth 0-**form** f, in which case $i_X f = 0$ and hence

$$i_X(df) + d(i_X f) = i_X\, df = df(X) = Xf = \mathcal{L}_X f.$$

(ii) Any smooth 1-form can be written locally as a sum of terms of the form $u\,dv$ for smooth functions u and v, so to prove (11) for 1-forms it suffices to consider the case $\omega = udv$.

– In this case, by Corollary 7,

$$\mathcal{L}_X(udv) = (\mathcal{L}_X u)dv + u(\mathcal{L}_X dv) = (Xu)dv + ud(Xv).$$

On the other hand,

$$\begin{aligned} i_X d(udv) + d(i_X(udv)) =& i_X(du \wedge dv) + d(uXv) \\ =& (i_X du) \wedge dv - du \wedge (i_X dv) + ud(Xv) + (Xv)du \\ & \qquad (\because \ \text{the interior product is an antiderivation}) \\ =& (Xu)dv - (Xv)du + ud(Xv) + (Xv)du \\ =& (Xu)dv + ud(Xv). \end{aligned}$$

(iii) Now let $k > 1$ and assume (11) has been proved for forms of degree less than k. Let ω be an arbitrary smooth k-form, written in smooth local coordinates as

$$\omega = \sum_I{}' \omega^I dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Writing

$$\alpha = \omega_I dx^{i_1} \quad \text{and} \quad \beta = dx^{i_2} \wedge \cdots \wedge dx^{i_k},$$

we see that ω can be written as a sum of terms of the form $\alpha \wedge \beta$, where α is a smooth 1-form and β is a smooth $(k-1)$-form.

– For such a term, the induction hypothesis imply

$$\begin{aligned} \mathcal{L}_X(\alpha \wedge \beta) =& (\mathcal{L}_X\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_X\beta) \\ =& (i_x d\alpha + d i_X \alpha) \wedge \beta + \alpha \wedge (i_X d\beta + d i_X \beta). \end{aligned} \tag{12}$$

On the other hand, using the fact that both d and i_X are antiderivations, we compute

$$\begin{aligned} i_X d(\alpha \wedge \beta) + d(i_X(\alpha\wedge\beta)) =& i_X(d\alpha \wedge \beta - \alpha \wedge d\beta) + d(i_X\alpha \wedge \beta - \alpha \wedge i_X\beta) \\ =& (i_X d\alpha) \wedge \beta + d\alpha \wedge (i_X\beta) \\ & - (i_X\alpha) \wedge d\beta + \alpha \wedge (i_X d\beta) \\ & + d(i_X\alpha) \wedge \beta + (i_X\alpha) \wedge d\beta \\ & - d\alpha \wedge (i_X\beta) + \alpha \wedge (i_X\beta) \\ =& (i_X d\alpha) \wedge \beta + \alpha \wedge (i_X d\beta) + d(i_X\alpha) \wedge \beta + \alpha \wedge (d i_X\beta). \quad \square \end{aligned}$$

Corollary 8 (The Lie Derivative Commutes with d). *If X is a smooth vector field and ω is a smooth differential form, then*

$$\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega).$$

Proof. By Cartan's formula and the fact that $d \circ d = 0$,

$$\mathcal{L}_X(d\omega) = i_X d(d\omega) + d(i_X d\omega) = d(i_X d\omega);$$

$$d(\mathcal{L}_X\omega) = d(i_X d\omega) + d(d(i_X\omega)) = d(i_X d\omega). \qquad \square$$