1. Complex and Almost-Complex Manifolds

Definition. *A complex structure on a real vector space V is a linear endomorphism J of V such that $J^2 = -1$, where 1 is the identity transformation of V.*

- A real vector space with a complex structure can be given the structure of a complex vector space.
  We define scalar multiplication by complex numbers as

$$(a + ib)X = aX + bJX, \quad \forall X \in V \quad \text{and} \quad a, b \in \mathbb{R}.$$

  Evidently thedimension m of V must be even, and we define its complex dimension to be $m/2$.

- Conversely, if we are given a complex vector space V of complex dimension n, we can define a linear endomorphism J of V by

$$J(X) = iX, \quad \forall X \in V.$$

  Then V, considered as a real vector space of dimension $2n$, has J as its complex structure.

- We now choose a special class of bases related to the complex structure.
  More precisely, we prove the following proposition.

Proposition 1. *Let J be a complex structure associated with a real $2n$ dimensional vector space V. Then there exist vectors $X_1, \cdots, X_n$ of V such that $\{X_1, \cdots, X_n, JX_1, \cdots, JX_n\}$ is a basis for V.*

Proof. Regard V as a complex vector space of n complex dimensions, for which we choose as a basis $X_1, \cdots, X_n$ of V.
- Then $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ form a basis for V as a real $2n$-dimensional space. □

- By setting

$$z^k = x^k + iy^k, \quad k = 1, \cdots, n,$$

  the complex vector space $\mathbb{C}^n$ can be identified with the real vector space $\mathbb{R}^{2n}$ of $2n$-tuples of real numbers $(x^1, \cdots, x^n, y^1, \cdots, y^n)$.
- The complex structure J induced from that of $\mathbb{C}^n$, maps $(x^1, \cdots, x^n, y^1, \cdots, y^n)$ into $(y^1, \cdots, y^n, -x^1, \cdots, -x^n)$ and is called the **canonical complex structure** of $\mathbb{R}^{2n}$.
- In terms of the natural (coordinate) basis of $\mathbb{R}^{2n}$, J is represented by the matrix

$$J_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \text{where } I_n \text{ is the identity matrix of degree } n.$$

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Proposition 2. *Let J, J' be complex structures on real vector spaces V and V', respectively.*
Let f be a real linear mapping of V into V'. Then, when V, V' are regarded as complex vector spaces, f is complex linear iff $J' \circ f = f \circ J$.

Proof. This follows trivially since J and J' are effectively multiplication by i when V and V' are regarded as complex vector spaces. □

Corollary 3. *Thus the complex linear group $GL(n;\mathbb{C})$ of degree n can be regarded as a subgroup of $GL(2n,\mathbb{R})$ consisting of matrices which commutes with the matrix J_0.*

- It follows that this representation of $\mathrm{GL}(n:\mathbb{C})$ into $\mathrm{GL}(2n:\mathbb{R})$, called the **real representation of GL**$(n,\mathbb{C})$, is given by

$$A + iB \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

  for $A + iB \in \mathrm{GL}(n,\mathbb{C})$.

Proposition 4. *Let J be a complex structure on the real vector space V and let V' be a (real) subspace of V.*
Then V' is invarianr under the action of J iff V' is a complex subspace of V when V is considered as a complex vector space.

Proof. This folloes because J means effectively multiplication by J. □

- Let V be a real vector space and let V^* denote its dual.
  Then a complex structure J on V induces a complex structure $\widetilde{J}$ on V^*; we define $\widetilde{J}$ by

$$X^*(JX) = \widetilde{J}X^*(X)$$

  for $X \in V$ and $X^* \in V^*$. We have

$$\widetilde{J}(\widetilde{J}X^*(X)) = \widetilde{J}X^*(JX) = X^*(J^2X) = -1(X^*(X))$$

  giving $\widetilde{J}^2 = -1$ as required. It will be convenient from now on to use the notation J instead of $\widetilde{J}$.

Complexification

- Let V be a vector space and consider the tensor product $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} C$.
  We can similarly form tensors of type (r,s) over starting with $V^{\mathbb{C}}$ instead of V, and then tensors of type (r,s) over V form a real subspace of tensors of type (r,s) over $V^{\mathbb{C}}$.
- The operation of complex conjugation in $V^{\mathbb{C}}$ is the real linear endomorphism defined by

$$Z = X + iY \to \overline{Z} = Z - iY, \quad \forall X, Y \in V.$$

  This endomorphism extends in a natural way to the space $T^r_s(V^{\mathbb{C}})$.

- Assume that V is a $2n$-dimensional real vecor space with a complex structure J. We can extend the action of J canonically to V^c and we also denote this extension by J which retains the property $J^2 = -1$.
  Thus the eigenvalues of this extended J are i and $-i$. Set

$$V^{1,0} = \{Z \in V^{\mathbb{C}} : JZ = iZ\}, \quad V^{0,1} = \{Z \in V^{\mathbb{C}} : JZ = -iZ\}$$

  Then

$$\begin{aligned} V^{1,0} =& \{X - iJX; X \in V\} \\ V^{0,1} =& \{X + iJX : X \in V\} \\ V^{\mathbb{C}} =& V^{1,0} \oplus V^{0,1}. \end{aligned}$$

  Moreover, complex conjugation defines a real linear isomorphism between $V^{1,0}$ and $V^{0,1}$.

- Obviously, the complexification ofthe dual space $(V^*)^{\mathbb{C}}$ is the same space as $(V^{\mathbb{C}})^*$, the dual space of $V^{\mathbb{C}}$.
– Again with respect to the eigenvalues i and $-i$ of the complex structure on V^* we have a direct sum decomposition

$$V^* = V_{1,0} \oplus V_{0,1};$$

  here

$$\begin{aligned} V_{1,0} =& \{X^* \in (V^*)^{\mathbb{C}}; X^*(X) = 0 \ \ \forall X \in V^{0,1}\} \\ V_{0,1} =& \{X^* \in (V^*)^{\mathbb{C}}; X^*(X) = 0 \ \ \forall X \in V^{1,0}\}. \end{aligned}$$

  The decomposition of $V^{\mathbb{C}}$ leads to the decomposition of the tensor space $T^r_s(V^{\mathbb{C}})$ into a direct sum of tensor products of vector spaces identical to one of the spaces $V^{1,0}$, $V^{0,1}$, $V_{1,0}$ and $V_{0,1}$.

- In particulat the exterior algebra $\Lambda V^{*\mathbb{C}}$ decomposes, and the exterior algebras $\Lambda V_{1,0}$ and ΛV^{*C} are subalgebras of $\Lambda V^{*\mathbb{C}}$.
– If we denote by $\Lambda^{p,q} V^{*\mathbb{C}}$ the subspace spanned by $\alpha \wedge \beta$ where $\alpha \in \Lambda^p V_{1,0}$ and $\beta \in \Lambda^q_{0,1}$, then

$$\Lambda V^{*\mathbb{C}} = \sum_{r=0}^{n} \Lambda^r V^{*\mathbb{C}} \ \text{ with } \Lambda^r V^{*\mathbb{C}} = \sum_{p+q=r} \Lambda^{p,q} V^{*\mathbb{C}}.$$

  Moreover complex conjugation in $V^{*\mathbb{C}}$ extends in a natural way to $\Lambda V^{*\mathbb{C}}$, and this gives a real linear isomorphism between $\Lambda^{p,q} V^{*\mathbb{C}}$ and $\Lambda^{q,p} V^{*\mathbb{C}}$.

2. Hermitian Metrics

Definiton. *A Hermitian metric is an inner product defined over a real vector space with complex structure J such that*

$$h(JX, JY) = h(X, Y) \quad \text{for all } X, Y \in V.$$

Lemma 5. *For an inner metric h, $h(X, JX) = 0$ for every vector X.*

Proof. $h(X, JX) = h(JX, J^2X) = -h(JX, X) = -h(X, JX)$. □

- This property gives rise to the following result.

Proposition 6. *Let h be a hermitian inner product in a real $2n$ dimensional vector space V with complex structure J. Then there exist vectors $X_1, \cdots, X_n$ of V such that $\{X_1, \cdots, X_n, JX_1, \cdots, JX_n\}$ is a orthonormal basis for V with respect to h.*

Proof. The proof is by induction on the dimension of V.

(i) We know that the claim is valid for $n = 1$ by using X_1 and JX_1.

(ii) Let W be the space spanned by X_1 and JX_1
and let $W^\perp$ be the orthogonal complement in V so that $V = W \oplus W^\perp$, and $W^\perp$ is invariant undet J.

– Now make use of the induction htpothesis that the proposition is true for $n - 1$, so that $W^\perp$ has an orthonormal basis $\{X_2, \cdots, X_n, JX_2, \cdots, JX_n\}$.
Then $[X_1, \cdots, X_n, JX_1, \cdots, JX_n]$ is the required basis. □

Proposition 7. *Let h be a hermitian inner product in a real $2n$ dimensional vector space V with complex structure J. Then h can be uniquely extended to a complex symmetric bilinear form, also denoted by h, of $V^{\mathbb{C}}$, such that*

$$\begin{aligned} h(\overline{Z}, \overline{W}) =& \overline{h(Z, W)}, \\ h(Z, \overline{Z}) >& 0, \quad \forall Z \neq 0, \\ h(Z, \overline{W}) =& 0 \ \ \forall Z \in V^{1,0}, W \in V^{0,1}. \end{aligned}$$

Conversely, every complex symmetric bilinear form h on $V^{\mathbb{C}}$ satisfying these conditions is the natural extension of a Hermitian inner product of V.

Proof. For example, writine $Z = X + iY$, $W = U + iV$, we have

$$\begin{aligned} h(\overline{Z}, \overline{W}) =& h(X - iY, U - iV) = h(X, U) + h(Y, V) - ih(X, V) - ih(Y, U), \\ h(Z, W) =& h(X - iY, U - iV) = h(X, U) + h(Y, V) + ih(X, V) + ih(Y, U). \end{aligned}$$ □

- To each hermitian inner product h on a real vector space with complex structure J, there corresponda an elemeb=nt ϕ of $\Lambda^2 V^*$ such that

The **skew-symmetry** follows since

$$\phi(X,Y) = h(JX, J^2Y) = -h(JX,Y) = -\phi(Y,JX) = -\phi(Y,X).$$

Moreover ϕ **is invariant under** J since

$$\phi(JX,JY) = h(JX, J^2Y) = -h(Y,JX) = -\phi(Y,X) = \phi(X,Y).$$

Since Λ^2V^* can be considered as a subspace of $\Lambda^2V^{*\mathbb{C}}$, it follows that ϕ uniquely determines a skew-symmetric bilinear form on $V^{\mathbb{C}}$, which we still denote by ϕ. Then $\phi \in \Lambda^{1,1}V^{*\mathbb{C}}$.

Lemma 8. *Let $Z_1, \cdots, Z_n$ be a basis for $V^{1,0}$ over $\mathbb{C}$ and let $\{\omega^1, \cdots, \omega^n\}$ be a dual basis of $V_{0,1}$. We write*

$$h_{j\bar{k}} = h(Z_j, \overline{Z}_k), \quad \text{for } j,k = 1, \cdots, n$$

Then

$$h_{j\bar{k}} = \overline{h_{k\bar{j}}}, \quad \text{for } j,k = 1, \cdots, n.$$

$$\phi(Z,W) = -i \sum_{j,k=1}^{n} h_{j\bar{k}}(\omega^j(Z)\overline{\omega}^k(W) - \omega^j(W)\overline{\omega}^k(Z)).$$

Lemma 9. *There is a natural $1-1$ correspondence between the set of complex structures on $\mathbb{R}^{2n}$ and the homogeneous space $\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$;*
the coset represented by an element $T \in \mathrm{GL}(2n:\mathbb{R})$ corresponds to the complex structure TJ_0T^{-1}, where J_0 is the canonical complex structure.

Proof. Note that every element $T \in \mathrm{GL}(2n;\mathbb{R})$ sends every complex structure J of $\mathbb{R}^{2n}$ into a complex structure TJT^{-1} of $\mathbb{R}^{2n}$.

– Consider $\mathrm{GL}(2n;\mathbb{R})$ as a group of transformations acting on the set of complex structures of $\mathbb{R}^{2n}$.
  It suffices to **claim**: this action is **transitive** and the subgroup which leaves J_0 invariant is $\mathrm{GL}(n;\mathbb{C})$.
– Indeed, let J and J' be two complex structures of $\mathbb{R}^{2n}$, and let

$$\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\} \ \text{ and } \ \{e'_1, \cdots, e'_n, J'e'_1, \cdots, J'e'_n\}.$$

  be bases of $\mathbb{R}^{2n}$. Define an element T of $\mathrm{GL}(2n;\mathbb{R})$ by

$$Te_k = e'_k, \ \ TJe_k = J'e'_k, \ \ \text{for } k = 1, \cdots, n.$$

  Then $J' = TJT^{-1}$, proving that the group acts transitively.
– Moreover, an element T of $\mathrm{GL}(2n:\mathbb{R})$ belongs to $\mathrm{GL}(n:\mathbb{C})$ iff it commutes with J_0, that is, $J_0 = TJ_0T^{-1}$. □

Lemma 10. *,There is a natural* $1-1$ *correspondence between the set of hermitian inner product in* $\mathbb{R}^{2n}$ *with respect to the complex structure* J_0 *and the homogeneous space* $\mathrm{GL}(2n, \mathbb{C})/\mathrm{U}(n)$.
The coset represented by an element $T \in \mathrm{GL}(n:\mathbb{C})$ *corresponds to the hermitian inner produch* h *defined by*

$$h(X,Y) = h_0(TX,TY), \quad \forall X,Y \in \mathbb{R}^{2n},$$

where h_0 *is the canonical hermitian inner product in* $\mathbb{R}^{2n}$.

Proof. Note that every element $T \in$GL$(n;\mathbb{C})$ sends a Hermitian inner product h (with respect to J_0) into a Hermitian inner product h' according to

$$h'(X,Y) = h(TX,TY), \quad \forall X,Y \in \mathbb{R}^{2n}.$$

- Consider $\mathrm{GL}(n;\mathbb{C})$ as a group of transformations acting on the set of hermitian inner product in $\mathbb{R}^{2n}$ with respect to J_0.
  It suffices to **claim**: this action is **transitive** and the subgroup which leaves h_0 invariant is $\mathrm{U}(n)$.
- We have already seen that there are orthogonal bases

  $$\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\} \text{ and } \{e'_1, \cdots, e'_n, J'e'_1, \cdots, J'e'_n\}$$

  with respect to h and h' in $\mathbb{R}^{2n}$.
- The element T of $\mathrm{GL}(2n;\mathbb{R})$ defined by

  $$Te'_k = e_k, \quad TJ_0e'_k = J_0e_k, \quad \text{for } k = 1, \cdots, n.$$

  is an element of $\mathrm{GL}(n;\mathbb{C})$ which sends h into h'.
- Moreove, the elements of $\mathrm{GL}(n;\mathbb{C})$ which leaves h_0 invariant is the intersection of $\mathrm{GL}(n;\mathbb{C})$ and $\mathrm{O}(2n)$, both being considered as subgroups of $\mathrm{GL}(2n;\mathbb{R})$.
  But these elements are precisely those of $\mathrm{U}(n)$, considered as a subgroup of $\mathrm{GL}(2n;\mathbb{R})$. □

3. Almost Complex Manifolds

Definiton. *An* **almost complex structure** *on a real differentiable manifold M is a tensor field J which at each point $x \in M$ is an endomorphism of the tangent space T_xM such that $J^2 = -1$, where 1 denotes the identity transformation of T_xM.*
A manifold with such a structure is called an **almost-complex manifold**.

Lemma 11. *An almost-complex manifold must be orientable.*

Lemma 12. *If $A \in M(n;\mathbb{C})$, then $\det_{\mathbb{R}} A = |\det_{\mathbb{C}} A|^2$.*

Proof of Lemma 11. Fix a basis $X_1, \cdots, X_n, JX_1, \cdots, JX_n$ in each T_xM.

- By Lemmaa 12, any two such bases differ by a linear transformation with positive determinant.
- To fix an orientation on M we consider the family of all coordinate systems $x^1, \cdots, x^{2n}$ of M such that in each coordinate neighborhood, the coordinate basis
$$\left(\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^{2n}}\right)$$
of T_xM at x differ from the chosen basis $(X_1, \cdots, X_n, JX_1, \cdots, JX_n)$ by a linear transformation of positive determinant.
These coordinate systems determine a complete atlas for M, which is thus oriented. □

- However, an even-dimensional, oriented real manifold does not necessarily have an almost complex structure.
For example, Ehresmann and Hopf proved that the 4-sphere $\mathbb{S}^4$ cannot have an almost complex structure.

Definition. *A complex manifold is a paracompact Hausdorff space which has a covering by neighborhoods each homeomorphic to an open set in $\mathbb{C}^n$ such that when two neighborhoods overlap the local coordinates transform by a* **holomorphic** *transformation.*
In other words, if $z^1, \cdots, z^n$ are local coordinates in one such neighborhood, and $w^1, \cdots, w^n$ are local coordinates of another neighborhood, then when these are both defined, we have
$$w^i = f^i(z^1, \cdots, z^n),$$
where each f^i is a holomorphic function of the z's, and the function determinant
$$\frac{\partial(w^1, \cdots, w^n)}{\partial(z^1, \cdots, z^n)} \neq 0.$$

- We proceed to prove the following.

Proposition 13. *A complex manifold admits an almost complex structure.*

Proof. We consider first the space $\mathbb{C}^n$ of n-tuples of complex numbers $(z^1, \cdots, z^n)$ with $z^k = x^k + iy^k$, $k = 1, \cdots, n$.
With respect to the coordinate system $x^1, \cdots, x^n, y^1, \cdots, y^n$, we define an almost complex structure J on $\mathbb{C}^n$ by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i}, \quad i = 1, \cdots, n$$

Lemma 14. *A mapping f of an open set of $\mathbb{C}^n$ into $\mathbb{C}^m$ preserves the almost-complex structure J defined in the above manner, that is $f_* \circ J = J \circ f_*$ iff f is holomorphic.*

Proof of Lemma 14. Let $(w^1, \cdots, w^m)$ be the natural coordinate system of $\mathbb{C}^m$ with $w^k = u^k + i\,v^k$. Then the map f is defined locally by

$$u^k = u^k(x^1, \cdots, x^n, y^1, \cdots, y^n) \quad v^k = v^k(x^1, \cdots, x^n, y^1, \cdots, y^n), \quad k = 1, \cdots, m.$$

The mapping f is holomorphic iff the following Cauchy-Riemann equarions are satisfied:

$$\frac{\partial u^k}{\partial x^j} = \frac{\partial v^k}{\partial y^j}, \quad \frac{\partial u^k}{\partial y^j} = -\frac{\partial v^k}{\partial x^j},$$

with $j = 1, \cdots, n$, $k = 1, \cdots, m$. However, for any f we have

$$\begin{aligned} f_*\left(\frac{\partial}{\partial x^j}\right) &= \sum_{k=1}^m \left(\frac{\partial u^k}{\partial x^j}\right)\frac{\partial}{\partial u^k} + \sum_{k=1}^m \left(\frac{\partial v^k}{\partial x^j}\right)\frac{\partial}{\partial v^k}, \\ f_*\left(\frac{\partial}{\partial y^j}\right) &= \sum_{k=1}^m \left(\frac{\partial u^k}{\partial y^j}\right)\frac{\partial}{\partial u^k} + \sum_{k=1}^m \left(\frac{\partial v^k}{\partial y^j}\right)\frac{\partial}{\partial v^k}, \end{aligned}$$

We have

$$\begin{aligned} f_* \circ J\left(\frac{\partial}{\partial x^j}\right) &= f_*\left(\frac{\partial}{\partial y^j}\right) \\ J \circ f_*\left(\frac{\partial}{\partial x^j}\right) &= \sum_{k=1}^m \left(\frac{\partial u^k}{\partial x^j}\right)\frac{\partial}{\partial v^k} - \sum_{k=1}^m \left(\frac{\partial v^k}{\partial x^j}\right)\frac{\partial}{\partial u^k}. \end{aligned}$$

From the Cauchy-Riemann equations, we have

$$f_* \circ J\left(\frac{\partial}{\partial x^j}\right) = J \circ f_*\left(\frac{\partial}{\partial x^j}\right).$$

Similarly we have

$$f_* \circ J\left(\frac{\partial}{\partial y^i}\right) = J \circ f_*\left(\frac{\partial}{\partial y^i}\right). \quad \square$$

Proof of Proposition 13 (continued). . To define an almost-complex structure on a complex manifold we transfer the almost-complex structure on $\mathbb{C}^m$ to M by means of the charts.

- This can be done unambiguously, for, by the previous result, the process is independent of the particular charts chosen. □

We have shown that a complex manifold admits an almost complex structure and therefore it must be orientable. However, it is interesting to given an independent proof as follows.

An Alernate Proof of That A Complex Manifold Must Be Orientable.

– Let $z^1, \cdots, z^m$ be a system of local coordinates on the m-dimensional complex manifold M.
  Set $z^k = x^k + iy^k$ so that x^k and y^k form a real system of local coordinates for the real $2m$-dimensional manifold. We write

$$x^k = \frac{1}{2}(z^k + \overline{z}^k), \quad y^k = -\frac{i}{2}(z^k - \overline{z}^k).$$

  Then we see that

$$dx^1 \wedge \cdots \wedge dx^m \wedge dy^1 \wedge dy^m = (-\frac{i}{2})^m dz^1 \wedge \cdots \wedge dz^m \wedge d\overline{z}^1 \wedge d\overline{z}^m$$

  Thus the form

$$\Theta(z) = (-\frac{i}{2})^m dz^1 \wedge \cdots \wedge dz^m \wedge d\overline{z}^1 \wedge d\overline{z}^m$$

  is a real form of maximal order $2m$.

– **Claim:** $\Theta(z)$ is well-defined up to a positive factor.
  Let $w^1, \cdots, w^m$ be another system of local coordinates. Then

$$dw^1 \wedge \cdots \wedge dw^m = \mathcal{D} dz^1 \wedge dz^m$$

  where $\mathcal{D} = \det \partial(w^1, \cdots, w^m)/\partial(z^1, \cdots, z^m)$. Then

$$d\bar{w}^1 \wedge \cdots \wedge d\bar{w}^m = \mathcal{D} d\bar{z}^1 \wedge d\bar{z}^m$$

  Hence

$$\Theta(w) = \mathcal{D}\overline{\mathcal{D}}\Theta(z)$$

  and our claim is justified.

– **To define Θ globally**, we choose a locally finite coverong and a partition of unity subordinate to the covering.
  This gives rise to a globally defined $2m$ form over M and hence M is orientable.

- We have seen that a vector space carrying a complex structure determines a splitting of the complexified space into the direct sum of vectors of type $(1,0)$ and those of type $(0,1)$.
- Conversely, a decomposition of this nature detemines a complex structure on the original vector space.

- An almost-complex structure determines vector fields of type (1,0) and (0,1), and dually it determines 1-forms of type $(1,0)$ and $(0,1)$.
- Conversely, if we are given the space of complex-valued 1-forms $T_{1,0}$ of type $(1,0)$,
  and if $T_{0,1}$ denotes the space of forms which are conjugate complex to those of $T_{1,0}$ so that we have the decomposition

$$T_x^{*\mathbb{C}} = T_{1,0} \oplus T_{0,1}$$

  then this determines the almost complex structure on the manifold.

An Alternate Proof of That A Complex Manifold Determines an Almost-Complex Structure.

– For on a complex-manifold the complex-valued 1-forms expressed in terms of local coordinates z^α are linear combinations of dz^α.
  We definethese to be the space of forms of type $(1,0)$, and the space of forms of type $(0,1)$ are similarly formed with $d\overline{z}^\alpha$.
  Since dz^α, $d\overline{z}^\alpha$ are linearly independent, they define an almost-complex structure.
– To describe J in terms of local coordinates, let $z^\alpha = x^\alpha + iy^\alpha$. We have

$$dz^\alpha\left(\frac{\partial}{\partial x^\beta}\right) = \delta^\alpha_\beta, \quad dz^\alpha\left(\frac{\partial}{\partial y^\beta}\right) = i\delta^\alpha_\beta.$$

  Then, using the fact that dz^α is of type $(1,0)$ we obtain

$$dz^\alpha\left(\frac{J\partial}{\partial x^\beta}\right) = i\,\delta^\alpha_\beta, \quad dz^\alpha\left(\frac{J\partial}{\partial y^\beta}\right) = -\delta^\alpha_\beta.$$

  These relations give

$$J\left(\frac{\partial}{\partial x^\beta}\right) = \frac{\partial}{\partial y^\beta}, \quad J\left(\frac{\partial}{\partial y^\beta}\right) = -\frac{\partial}{\partial x^\beta},$$

  and hence $J^2 = -1$ as it should.

- The **question** arises naturally whether all almost-complex structures arise only from complex-manifolds

—more precisely, **whether every almost-complex manifold is complex**.

- This indeed is the case for real dimension 2 but not in general.
- We now find necessary conditions that an almost-complex manifold should arise from a complex manifold.

- Consider the differential operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right)(dx^j + i\,dy^j)$$

and

$$\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)(dx^j - i\,dy^j),$$

which operate on such a form

$$\omega(z) = \eta(z)dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}.$$

by

$$\partial\omega = \frac{\partial\eta}{\partial z^i}dz^i \wedge dz^{i_1} \wedge \cdots \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q},$$

and

$$\overline{\partial}\omega = \frac{\partial\eta}{\partial\overline{z}^j}d\overline{z}^j \wedge dz^{i_1} \wedge \cdots \wedge d\overline{z}^{j_1} \wedge \cdots \wedge dz^{\overline{j}_q}.$$

Lemma 15. *On a complex manifold, the following relations hold:*

$$d = \partial + \overline{\partial}, \quad \partial\partial = 0, \quad \overline{\partial\partial} = 0.$$

– These relations need hold for an arbitrary, almost complex manifold,
but it is readily verified that **if one of them holds, so do the other two**.
In such a case, the almost complex structure is said to be **integrable**.
Thus, for a complex manifold, the almost complex structure is said to be **integrable**.

– The converse to this is known as the **Newlander-Nirenberg theorem**: any manifold with an integrable, almost complex structure has a holomorphic coordinate chart.