Covectors

Definition. *Let V be a finite-dimensional vector space. A* **covector** *on V is real-valued linear functional on V, that is, a linear map $\omega : V \to \mathbb{R}$.*

- The space of all covectors on V is itself a real vector space under the obvious operations of pairwise addition and scalar multiplication.

— It is denoted by V^* and called the **dual space** to V.

Proposition 6.1. *Let V be a finite-dimensional vector space. If $(E_1, \cdots, E_n)$ is any basis for V, then the covectors $(\varepsilon^1, \cdots, \varepsilon^n)$ defined by*

$$\varepsilon^i(E_j) = \delta^i_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

form a basis for V^*, called the **dual basis to** (E_i)*. Therefore, $\dim V^* = \dim V$.*

Example. If (e_i) denote the standard basis for $\mathbb{R}^n$, we denote the dual basus by

$$(e^1, \cdots, e^n) \quad \text{(note the upper indices)},$$

and call it the **standard dual basis**.

— The basis vectors are the linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ given by

$$e^j(v) = e^j(v^1, \cdots, v^n) = v^j.$$

— In other words, e^j is just the linear functional that picks out the jth component of a vector.

— In matrix notation, a linear map from $\mathbb{R}^n$ to $\mathbb{R}$ is represented by a $1 \times n$ matrix; i.e. a row matrix.

– The basis covectors can therefore also be thought of as the linear functionals represented by the row matrix

$$e^1 = (1 \;\; 0 \cdots 0), \cdots, e^n = (0 \cdots 0 \;\; 1).$$

- If (E_i) is a basis for V and (ε^j) is its dual basis, then for any matrix $X = X^i E_i \in V$, we have

$$\varepsilon^j(X) = X^i \varepsilon^j(E_i) = X^i \delta^j_i = X^j.$$

Thus as in the case of $\mathbb{R}^n$, ε^j picks out the jth component of a vector w.r.t. the basis (E_i).

— More generally, Proposition 6.1 shows that we can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_j \varepsilon^j, \tag{6.1}$$

where the components ω_j are determined by

$$\omega_j = \omega(E_j). \tag{6.2}$$

- We will write basis covectors with upper indices, and components of a covector with lower indices, because this helps to ensure that mathematically meaningful expressions such as (6.1) will always follow our index conventions:
  Any index that is to be summed over in a given term appears exactly twice, once as a subscript and once as a superscript.

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Proposition 6.2. *The dual map satisfies the following properties.*
(a) $(A \circ B)^* = B^* \circ A^*$.
(b) $(\mathrm{Id}_V)^*; V^* \to V^*$ *is the identity map of* V^*.

- Apart from the fact that $\dim V^* = \dim V$, the second important fact is the following characterization of the **second dual space** $V^{**} = (V^*)^*$.
- For each vector space V there is a natural basis-independent map $\xi : V \to V^{**}$, defined as follows.
  For each vector $X \in V$, define a linear functional $\xi(X) : V^* \to \mathbb{R}$ by

$$\xi(X)(\omega) = \omega(X), \quad \forall \omega \in V^*.$$

Proposition 6.4. *For a finite-dimensional vector space* V, *the map* $\xi : N \to V^{**}$ *is an isomorphism.*

Proof. Since V and V^* have the same dimension, it suffices to **claim:** ξ **is injective**.
Indeed, suppose $X \in V$ is not zero. Extend X to a basis $(E_1, E_2, \cdots, E_n)$ for V, $X = E_1$ and let $(\varepsilon^1, \cdots, \varepsilon^n)$ denote the dual basis for V^*. Then

$$\xi(X)(\varepsilon^1) = \varepsilon^1(X) = \varepsilon^1(E_1) = 1 \neq 0,$$

so $\xi(X) \neq 0$. □

- The preceeding proposition shows that when V is finite-dimensional, we can unambiguously identify V^{**} with V itself, because the map ξ is canonically defined, without reference to any basis.
- It is important to observe that although V^* is also isomorphic to V, there is **no canonical** isomorphism $V \cong V^*$.

- Because of Proposition 6.4, the real number $\omega(X)$ obtained by applying a covector ω to a vector X is sometimes denoted by either of the more symmetric-looking notations

$$\langle \omega, X \rangle, \quad \langle X, \omega \rangle;$$

  both expressions can be thought of either as the action of the covector $\omega \in V^*$ on the vector $X \in V$, or as the action of the covector $\xi(X) \in V^{**}$ on the element $\omega \in V^*$.
- Whenever one of the arguments is a vector and the other a covector, the notation $\langle \omega, X \rangle$ is always to be interpreted as the actual pairing between vectors and covectors, not as an inner product.

Tangent Covectors on Manifolds

Definition. *Let M be a smooth manifold. For each $p \in M$, we define the* **cotagent space** *at p, denoted by T_p^*M, to be the dual space to T_pM:*

$$T_p^*M = (T_pM)^*.$$

Elements of T_p^*M are called **tangent covectors** *at p.*

- If (x^i) are smooth local coordinates on an open subset $U \subset M$, then $\forall p \in U$, the coordinate basis $(\frac{\partial}{\partial x^i}\Big|_p)$ give rise to a dual basis for T_p^*M, which we denote for the moment by $(\lambda^i\Big|_p)$.
- Any covector $\omega \in T_p^*M$ can thus be written uniquely as $\omega = \omega_i \lambda^i\Big|_p$, where

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

- Let $(\widetilde{x}^j)$ be another set of smooth local coordinates whose domain contains p, and let $(\widetilde{\lambda}^i\Big|_p)$ denote the basis for T_p^*M dual to $(\frac{\partial}{\partial \widetilde{x}^j}\Big|_p)$.
- We can compute the components of the same covector ω w.r.t. the new coordinate system as follows.

— First recall that the coordinate vector fields transform as follows:

$$\frac{\partial}{\partial x^i}\bigg|_p = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p. \tag{6.4}$$

— Writing ω in both systems as

$$\omega = \omega_i \lambda^i\Big|_p = \widetilde{\omega}_j \lambda^j\Big|_p,$$

we can use (6.4) to compute the components ω_i in terms of $\widetilde{\omega}_j$:

$$\omega_i = \omega\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \omega\left(\frac{\partial \widetilde{x}^j}{\partial x^i}(p)\frac{\partial}{\partial \widetilde{x}^j}\bigg|_p\right) = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\widetilde{\omega}_j. \tag{6.5}$$

- In the early days of smooth manifold theory, before most of the abstract coordinate-free definition we are using were developed, mathematicians tended to think of a tangent vector at a point p as an assignment of an n-tuples $(X^1, \cdots, X^n)$ and $(\widetilde{X}^1, \cdots, \widetilde{X}^n)$ assigned to two different systems (x^i) and $(\widetilde{x}^j)$ were related by the transformation law:

$$\widetilde{X}^j = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)X^i. \tag{6.6}$$

- Similarly, a tangent covector was thought of as n-tuple $(\omega_1, \cdots, \omega_n)$ that transform, by virtue of (6.5), according to the following slightly different rule:

$$\omega_i = \frac{\partial \widetilde{x}^j}{\partial x^i}(p)\widetilde{\omega}_j. \tag{6.7}$$

- Thus it becomes customary to call the tangent covectors **covariant vectors** because their components transform in the same way as ("vary with") the coordinate partial derivatives, with the Jacobian matrix $(\frac{\partial \widetilde{x}^j}{\partial x^i})$ multiplying the objects associated with the "new" coordinates $(\widetilde{x}^j)$ to obtain those associated with the "old" coordinates (x^i).

— Analogously, tangent vectors were called **contravariant vectors**, because their components transform in the opposite way.

The Cotangent Bundle

Definition. *The disjoint union*

$$T^*M = \coprod_{p \in M} T_p^*M$$

is called the **cotangent bundle** *of M. It has a natural projection map*

$$\pi : T^*M \to M$$

sending $\omega \in T^*M$ to $p \in M$.

- As above, given any smooth local coordinates (x^i) on $U \subset M$, for each $p \in U$ we denote the basis for T_p^*M dual to $\frac{\partial}{\partial x^i}\Big|_p$ by $(\lambda^i\Big|_p)$.

  This defines n maps $\lambda^1, \cdots, \lambda^n : U \to T^*M$, called **coordinate covector fields**.

Proposition 6.5. *Let M be a smooth manifold and let T^*M be its cotangent bundle. With the standard projection map and the natural vector space structure on each fiber, T^*M has a unique smooth manifold structure making it into a rank-n vector bundle over M for which all coordinate covector fields are smooth local sections.*

Proof. Given a smooth chart (U, φ) on M, with coordinate functions (x^i), define $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi(\xi_i \lambda^i\Big|_p) = (p, (\xi_1, \cdots, \xi_n)),$$

where λ^i is the ith coordinate covector field associated with (x^i).

— Suppose $(\widetilde{U}, \widetilde{\varphi})$ is another smooth chart on M, with coordinate functions $(\widetilde{x}^i)$, and let $\widetilde{\Phi} : \pi^{-1}(\widetilde{U}) \to \widetilde{U} \times \mathbb{R}^n$ be defined analogously.

— On $\pi^{-1}(U \cap V)$, it follows from (6.5) that

$$\Phi \circ \widetilde{\Phi}^{-1}(p, (\widetilde{\xi}_1, \cdots, \widetilde{\xi}_n)) = \left(p, \left(\frac{\partial \widetilde{x}^j}{\partial x^1}(p)\widetilde{\xi}_j, \cdots, \frac{\partial \widetilde{x}^j}{\partial x^n}(p)\widetilde{\xi}_j\right)\right).$$

The $\mathrm{GL}(n, \mathbb{R})$-valued function $(\frac{\partial \widetilde{x}^j}{\partial x^i})(p)$ is smooth, so it follows that T^*M has a smooth structure making it into a smooth vector bundle for which the map Φ are smooth local trivilizations. $\square$

- As in the case of the tangent bundle, smooth local coordinates for M yield smooth local coordinates for its cotangent bundle.
- If (x^i) are smooth coordinates on an open set $U \subset M$, then the map $\pi^{-1}(U)$ to $\mathbb{R}^{2n}$ given by

$$\xi_i \lambda^i\Big|_i \mapsto (x^1(p), \cdots, x^n(p), \xi_1, \cdots, \xi_n)$$

  is a smooth coordinate chart for T^*M.

— We will call (x^i, ξ_i) the **standard coordinates** for T^*M associated with (x^i).

Definition. *A section of T^*M is called a* **covector field** *or a* **(differential) 1-form**.

- In any smooth local coordinates on an open set $U \subset M$, a covector field ω can be written in terms of the coordinate covector fields (λ^i) as $\omega = \omega_i \lambda^i$ for n functions $\omega_i : U \to \mathbb{R}$ called the **component functions** of ω.
  They are characterized by

$$\omega_i(p) = \omega_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right).$$

Lemma 6.6 (Smoothness Criteria for Covector Fields). *Let M be a smooth manifold, and let $\omega : M \to T^*M$ be a rough section.*

(1) *If $\omega = \omega_i \lambda^i$ is the coordinate representation for ω in any smooth chart (U, x^i) for M, then ω is smooth iff its components functions are smooth.*

(2) *ω is smooth iff for every vector field X on an open subset $U \subset M$, the function $\langle \omega, X\rangle : U \to \mathbb{R}$ defined by*

$$\langle \omega, X\rangle(p) = \langle \omega_p, X_p\rangle = \omega_p(X_p)$$

is smooth.

Definition. *We denote the real vector space of all smooth covector fields on M by $\mathcal{T}^*(M)$.*

— As smooth sections of a vector bundle, elements of $\mathcal{T}^*(M)$ can be multiplied by smooth real-valued function:
  If $f \in C^\infty(M)$ and $\omega \in \mathcal{T}^*(M)$, the covector field $f\omega$ is defined by

$$(f\omega)_p = f(p)\omega_p. \tag{6.8}$$

— Like the space of smooth vector fields, $\mathcal{T}^*(M)$ is a module over $C^\infty(M)$.

- Geometrically, we think of a vector field on M as a rule that attaches an arrow to each point of M.
- **What kind of geometric picture can we form of a covector field?**

— The key idea is that a nonzero linear functional $\omega_p \in T_p^*M$ is completely determined by two pieces of data:

  (1) its kernel, which is a codimension-1 linear subspace of T_pM (a hyperplane), and
  (2) the set of vectors X for which $\omega_p(X) = 1$, which is an affine hyperplane parallel to the kernel.

- Thus you can **visualize a covector field** as defining **a pair of affine hyperplanes** in each tangent space,
  one through the origin and another parallel to it, and varying continuously from point to point.

— At points where the covector field takes on the value zero, one of the hyperplanes goes off to infinity.

The Differential of a Function

- In elementary calculus, the gradient of a smooth real-valued function f on $\mathbb{R}^n$ is defined as the vactor field whose components are the partial derivatives of f. In our notation, this would read

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

  Unfortunately, in this form, the gradient does **not** make **coordinate independent** sense.

Example. Let $f(x, y) = x^2$ on $\mathbb{R}^2$, and let X be the vctor field

$$X = \operatorname{grad} f = 2x \frac{\partial}{\partial x}.$$

Compute the coordinate expression of X in polar coordinates of X in polar coordinates (on some open set on which they are defined) and show that $\operatorname{grad} f$ is not equal to

$$\frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}.$$

- Although the first partial derivatives of a smooth function cannot be interpreted in a coordinate-independent way as the components of a vector field, it turns out that they **can** be interpreted as the components of a covector field.
- This is the most important application of covector fields.

Definition. *Let f be a smooth real-valued function on a smooth manifold M. We define a covector field df, called the* **differential** *of f, by*

$$df_p(X_p) = X_p f, \quad \forall X_p \in T_p M.$$

Lemma 6.7. *The differential of a smooth function is a smooth covector field.*

Proof. (**1**) It is straightforward to verify that at each point $p \in M$, $df_p(X_p)$ depends linearly on X_p, so that df_p is indeed a covector at p.

(2) To see that df is **smooth**, we use Lemma 6.6 (b).
For any smooth vector field X on an open subset $U \subset M$, the function $\langle df, X \rangle$ is smooth because it is equal to Xf. □

- **To see what df looks like more concretely**, we need to compute its coordinate representations.

— Let (x^i) be smooth coordinates on an open subset $U \subset M$, and let (λ^i) be the corresponding coordinate coframe on U.

— Writing df in coordinates as $df_p = A_i(p)\lambda^i\Big|_p$ for some functions $A_i : U \to \mathbb{R}$, the definition of df implies

$$A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p).$$

– This yields the following formula for the coordinate representation of df:

$$df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i\bigg|_p. \tag{6.9}$$

– Thus the component functions of df in any smooth coordinate chart are the partial derivatives of f w.r.t. those coordinates.

– Because of this, we can think of df as an anologue of the classical gradient, reinterpreted in a way that makes **coordinate-independent** sense on a manifold.

- If we apply (6.9) to the special case in which f is one of the coordinate functions $x^j : U \to \mathbb{R}$, we obtain

$$dx^j\Big|_p = \frac{\partial x^j}{\partial x^i}\lambda^i\Big|_p = \delta^j_i\lambda^i\Big|_p = \lambda^j\Big|_p.$$

In other words, **the coordinate covector field λ^j is dx^j**. Therefore, (6.9) can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p)dx^i\bigg|_p,$$

or as an equation between covector fields instead of covectors:

$$df = \frac{\partial f}{\partial x^i}dx^i. \tag{6.10}$$

In particular, in the 1-dimensional case, this reduces to

$$df = \frac{df}{dx}dx.$$

Example 6.8. If $f(x, y) = x^2 y \cos x$ on $\mathbb{R}^2$, then

$$\begin{aligned} df &= \frac{\partial(x^2 y\cos x)}{\partial x}dx + \frac{\partial(x^2 y\cos x)}{\partial y}dy \\ &= (2xy\cos x - x^2 y\sin x)dx + (x^2\cos x)dy. \end{aligned}$$

- It is important to observe that for a smooth real-valued function $f : M \to \mathbb{R}$, we have now defined two different kinds of derivatives of f at a point $p \in M$.
  (1) The pushforward f_* is defined as a linear map from T_pM to $T_{f(p)}\mathbb{R}$.
  (2) The differential df_p as a covector at p; i.e., a linear map from T_pM to $\mathbb{R}$.

These are really the same at p, once we take into account the canonical identification between $\mathbb{R}$ and its tangent space at any point; one easy way to see this is to note that both are represented in coordinates by the row matrix whose components are the partial derivatives of f.

Proposition 6.11 (Derivative of a Function Along a Curve). *Suppose M is a smoth manifold, $\gamma : M \to \mathbb{R}$ is a smooth curve, and $f : M \to \mathbb{R}$ is a smooth function. Then the derivative of the real-valued function $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is given by*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}(\gamma'(t)).$$

Proof. Directly from the definition, for any $t_0 \in J$,

$$\begin{aligned} df_{\gamma(t_0)}(\gamma'(t_0)) &= \gamma'(t_0)f \\ &= \left(\gamma_* \frac{d}{dt}\Big|_{t_0}\right) f \\ &= \frac{d}{dt}\Big|_{t_0}(f \circ \gamma) \\ &= (f \circ \gamma)'(t_0). \qquad \square \end{aligned}$$

- If γ is smooth curve in M, we have two different meanings for the expression $(f \circ \gamma)'(t)$.
  (1) $f \circ \gamma$ can be interpreted as a smooth curve in $\mathbb{R}$. Thus $(f \circ \gamma)'(t)$ is its tangent vector at the point $f \circ \gamma(t)$, an element of the tangent space $T_{f \circ \gamma(t)}\mathbb{R}$. This tangent vector is equal to $f_*(\gamma'(t))$.
  (2) $f \circ \gamma$ can also be considered simply as a real-valued function of one real variable, and then $(f \circ \gamma)'(t)$ is just its ordinary derivative. Proposition 6.11 shows that this derivative is equal to the real number $df_{\gamma(t)}(\gamma'(t))$.

Pullbacks

- Let $F : M \to N$ be a smooth map and $p \in M$ be arbitrary. The pushroward map
$$F_* : T_pM \to T_{F(p)}N$$
yields a dual linear map
$$F^* : T^*_{F(p)}N \to T^*_pM,$$
which is characterized by
$$(F^*\omega)(X) = \omega(F_*X), \quad \forall \omega \in T^*_{F(p)}N,\ X \in T_pM.$$
- When we introduced the pushforward map, we made a point of noting that vector fields do not pushforward to vector fields, except in the special case of a diffeomorphism.
- The surprising thing about pullbacks is that **smooth vector fields always pull back to smooth covector fields**.

Definition. *Given a smoth map $G : M \to N$ and a smooth covector field ω on N, define a covector field $G^*\omega$ on M by*

$$(G^*\omega)_p = G^*(\omega_{G(p)}). \tag{6.12}$$

- Observe that there is no ambiguity about what point to pull back from, in contrast to the vector field case.

Lemma 6.12. *Let $G : M \to N$ be a smooth map. Suppose $f \in C^\infty(N)$ and $\omega \in \mathcal{T}^*(N)$. Then*

$$G^*df = d(f \circ G); \tag{6.13}$$
$$G^*(f\omega) = (f \circ G)G^*\omega. \tag{6.14}$$

Proof. To prove (6.13), we let $X_p \in T_pM$ be arbitrary and compute

$$\begin{aligned}
(G^*df)_p(X_p) &= (G^*(df_{G(p)}))(X_p) && \text{(by (6.12))}\\
&= df_{G(p)}(G_*X_p) && \text{(by definition of } G^*)\\
&= (G_*X_p)f && \text{(by definition of } df)\\
&= X_p(f \circ G) && \text{(by definition of } G_*)\\
&= d(f \circ G)_p(X_p) && \text{(by definition of } d(f \circ G)).
\end{aligned}$$

Similarly, for (6.14) we compute

$$\begin{aligned}
(G^*(f\omega))_p &= (G^*(f\omega)_{G(p)}) && \text{(by (6.12))}\\
&= G^*(f(G(p))\omega_{G(p)}) && \text{(by (6.8))}\\
&= f(G(p))G^*(\omega_{G(p)}) && \text{(by linearity of } G^*)\\
&= f(G(p))(G^*\omega)_p && \text{(by (6.12))}\\
&= ((f \circ G)G^*\omega)_p && \text{(by (6.8)).} \qquad \square
\end{aligned}$$

Proposition 6.13. *Suppose $G : M \to N$ is smooth, and let ω be a smooth covector field on N. Then $G^*\omega$ is a smooth vector field on M.*

Proof. Let $p \in M$ be arbitrary, and choose smooth coordinates (x^i) for M near p and (y^j) for N near $G(p)$.
Writing ω in coordinates as $\omega = \omega_j dy^j$ for smooth functions ω_j defined near $G(p)$ and using Lemma 6.12 twice, we have the following computation in a nbhd of p:

$$G^*\omega = G^*(\omega_j dy^j) = (\omega_j \circ G)G^* dy^j = (\omega_j \circ G)d(y^j \circ G),$$

which is smooth. □

- In the couse of the preceeding proof we derived the following formula for the pullback of a covector field w.r.t. smooth coordinates (x^i) on the domain and (y^j) on the range:

  $$G^*\omega = G^*(\omega_j dy^j) = (\omega_j \circ G)d(y^j \circ G) = (\omega_j \circ G)dG^j, \tag{6.15}$$

  where G^j is the jth component function of G in these coordinates.
- In other words, to compute $G^*\omega$, all we need to do is to substitute the component functions of G for the coordinate functions of N everywhere that appear in ω.

Example. Let $G : \mathbb{R}^3 \to \mathbb{R}^2$ be the map given by

$$(u, v) = G(x, y, z) = (x^2 y, y \sin z),$$

and let $\omega \in \mathcal{T}^*(\mathbb{R}^2)$ be the covector field

$$\omega = u\, dv + v\, du.$$

According to (6.15), the pullback $G^*\omega$ is given by

$$\begin{aligned} G^*\omega =&(u \circ G)d(v \circ G) + (v \circ G)d(u \circ G) \\ =&(x^2)d(y \sin z) + (y \sin z)d(x^2 y) \\ =&x^2 y(\sin z\, dy + y \cos z\, dz) + y \sin z(2xy\, dx + x^2\, dy) \\ =&2x^2 y \sin z dx + 2x^2 y \sin z dy + x^2 y^2 \cos z dz. \end{aligned}$$

Example. Let (r, θ) be polar coordinates on the half-plane $H = \{(x, y) : x > 0\}$.
— We can think of the change of coordinates $(x, y) = (r \cos\theta, r \sin\theta)$ as the coordinate expression for the identity map on H,
but using (r, θ) as coordinates for the domain and (x, y) for the range.
— Then the pullback formula (6.15) tells us what we can compute the polar coordinate expression for a covector field simply by substituting $x = r\cos\theta$, $y = \sin\theta$ and expanding. For example,

$$\begin{aligned} x\, dy - y\, dx =& Id^*(x\, dy - y\, dx) \\ =&r \cos\theta d(r \sin\theta) - r \sin\theta d(r \cos\theta) \\ =&r \cos\theta(\sin\theta\, dr + r \cos\theta\, d\theta) - r \sin\theta(\cos\theta\, dr - r \sin\theta\, d\theta) \\ =&(r \cos\theta \sin\theta - r \sin\theta \cos\theta)dr + (r^2 \cos^2\theta + r^2 \sin^2\theta)d\theta \\ =&r^2 d\theta. \end{aligned}$$