

Integral Curves

Definition. If V is a smooth vector field on M , an **integral curve** of V is smooth curve $\gamma : J \rightarrow M$ such that

$$\gamma'(t) = V_{\gamma(t)} \quad \forall t \in J.$$

I.e., the tangent vector to γ at each point is equal to the value of V at that point. If $0 \in J$, the point $p = \gamma(0)$ is called the **starting point** of γ .

Example 1. (Integral Curves).

- (a) Let $V = \frac{\partial}{\partial x}$ be the first coordinate vector field on \mathbb{R}^2 .
- It is easy to check that the integral curves of V are precisely the straight lines parallel to the x -axis, with parametrizations of the form $\gamma(t) = (a + t, b)$ for constants a and b .
 - Thus there is a unique integral curve starting at each point of the plane, and the images of different curves are either identical or disjoint.
- (b) Let $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on \mathbb{R}^2 .
- If $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = (x(t), y(t))$, then the condition $\gamma'(t)$ for γ to be an integral curve translates to

$$x'(t) \frac{\partial}{\partial y} \Big|_{(x(t), y(t))} + y'(t) \frac{\partial}{\partial x} \Big|_{(x(t), y(t))} = x(t) \frac{\partial}{\partial y} \Big|_{(x(t), y(t))} - y(t) \frac{\partial}{\partial x} \Big|_{(x(t), y(t))}.$$

- Comparing the components of the vectors, we see that this is equivalent to the system of ODEs

$$\begin{cases} x'(t) = -y(t), \\ y'(t) = x(t). \end{cases}$$

These equations have the solutions

$$\begin{cases} x(t) = a \cos t - b \sin t, \\ y(t) = a \sin t + b \cos t, \end{cases}$$

for arbitrary constants a and b , and thus each curve of the form

$$\gamma(t) = (a \cos t - b \sin t, a \sin t + b \cos t)$$

is an integral curve of W .

- When $(a, b) = (0, 0)$, this is the constant curve $\gamma(t) = (0, 0)$; otherwise it is a curve traversed counterclockwise.
- Since $\gamma(0) = (a, b)$, we see once again that there is a unique integral curve starting at each point $(a, b) \in \mathbb{R}^2$, and the image of the various integral curves are either identical or disjoint.

- Writing γ in a smooth local coordinates as $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$, the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V can be written on a smooth coordinate domain $U \subset M$ as

$$(\gamma^i)'(t) \frac{\partial}{\partial x^i} \Big|_{\gamma(t)} = V^i(\gamma(t)) \frac{\partial}{\partial x^i} \Big|_{\gamma(t)},$$

which reduces to the system of ODEs

$$\begin{aligned} (\gamma^1)'(t) &= V^1(\gamma^1(t), \dots, \gamma^n(t)), \\ &\vdots \\ (\gamma^n)'(t) &= V^n(\gamma^1(t), \dots, \gamma^n(t)), \end{aligned}$$

where the component functions V^i are smooth on U .

Theorem 9. (ODE Existence, Uniqueness, and Smoothness). *Let $U \subset \mathbb{R}^n$ be open, and let $V : U \rightarrow \mathbb{R}^n$ be a smooth map.*

For $t_0 \in \mathbb{R}$ and $x \in U$, consider the following initial value problem.

$$(1) \quad \begin{cases} (\gamma^i)'(t) = V^i(\gamma(t)), \\ \gamma^i(t_0) = x^i. \end{cases}$$

- (a) **Existence:** $\forall t_0 \in \mathbb{R}$ and $x_0 \in U$, \exists an open interval J_0 containing t_0 and an open set $U_0 \subset U$ containing x_0 such that for each $x \in U_0$, there is a smooth curve $\gamma : J_0 \rightarrow U$ that solves (1).
- (b) **Uniqueness:** Any two differentiable solutions to (1) agree on their common domain.
- (c) **Smoothness:** Let t_0, x_0, J_0 and U_0 be as in (a), and define a map $\theta : J_0 \times U_0 \rightarrow U$ by letting $\theta(t, x) = \gamma(t)$, where $\gamma : J_0 \rightarrow U$ is the unique solution to (1) with initial condition x . Then θ is smooth.
 - There is a **unique** solution, at least for t in a small time interval $(-\epsilon, \epsilon)$, satisfying any initial condition of the form $(\gamma^1(0), \dots, \gamma^n(0)) = (a^1, \dots, a^n)$ for $(a^1, \dots, a^n) \in U$; the solution depends smoothly on both t and a .
 - This implies that there is a **unique** integral curve, at least for a short time, starting at any point in the manifold.
 - ⊙ Moreover, we will see that up to reparametrization, there is a unique integral curve passing through each point.
 - The following lemma shows how an integral curve can be reparametrized to change its starting point.

Translation Lemma. Let V be a smooth vector field on a smooth M , let $J \subset \mathbb{R}$ be an open interval, let $J \subset \mathbb{R}$, and let $\gamma : J \rightarrow M$ be an integral curve of V . For any $a \in \mathbb{R}$, let $J + a$ be interval

$$J + a = \{t + a : t \in J\}.$$

Then the curve $\tilde{\gamma} : J + a \rightarrow M$ defined by $\tilde{\gamma}(t) = \gamma(t - a)$ is an integral curve of V .

Proof. We can examine the action of $\tilde{\gamma}'(t)$ on a smooth real-valued function f defined in a nbhd of a point $\tilde{\gamma}(t_0)$. By the chain rule and the fact that γ is an integral curve,

$$\begin{aligned} \tilde{\gamma}'(t_0)f &= \frac{d}{dt} \Big|_{t=t_0} (f \circ \tilde{\gamma})(t) = \frac{d}{dt} \Big|_{t=t_0} (f \circ \gamma)(t - a) \\ &= \gamma'(t_0 - a)f = V_{\gamma(t_0 - a)}f = V_{\tilde{\gamma}(t_0)}f. \end{aligned}$$

Thus $\tilde{\gamma}$ is an integral curve of V . \square

Local And Global Flows

- Let V be a smooth vector field on a smooth manifold M with the property that $\forall p \in M$ there is a unique integral curve $\theta^{(p)} : \mathbb{R} \rightarrow M$ starting at p .
- (It may not always be the case that every integral curve is defined $\forall t \in \mathbb{R}$, but let us assume so for the time being).
- $\forall t \in \mathbb{R}$, we can define a map θ_t from M to itself by sending each point $p \in M$ to the point obtained by following the integral curve starting at p for time t :

$$\theta_t(p) = \theta^{(p)}(t).$$

This defines a family of maps $\theta_t : M \rightarrow M$ for $t \in \mathbb{R}$.

- Each such map “slides” the entire manifold along the integral curves for time t .
- If we set

$$q = \theta^{(p)}(s),$$

the translation lemma implies that $t \rightarrow \theta^{(p)}(t + s)$ is an integral curve starting at q ; since we are assuming uniqueness of integral curves, we must have $\theta^{(q)}(t) = \theta^{(p)}(t + s)$; that is

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p).$$

- Together with the equation $\theta_0(p) = \theta^{(p)}(0) = p$, which holds by definition, this implies that the map $\theta : \mathbb{R} \times M \rightarrow M$ is an action of the additive group \mathbb{R} on M .

Definition. The map $(t, p) \mapsto \theta^{(p)}(t)$ is called the **local flow** of the vector field V or a **local one-parameter group associated to V** . The curve $\theta^{(p)}(t)$ is called the **integral curve** of V through p .

Definition. Define a **global flow** on M (sometimes also called a **one-parameter group action**) to be a left action of \mathbb{R} on M ; that is, a continuous map $\theta : \mathbb{R} \times M \rightarrow M$ satisfying the following properties $\forall s, t \in \mathbb{R}$ and $\forall p \in M$:

$$\theta(t, \theta(s, p)) = \theta(t + s, p), \quad \theta(0, p) = p.$$

- Given a global flow θ on M , we define two collections of maps as follows:

(1) $\forall t \in \mathbb{R}$, define $\theta_t : M \rightarrow M$ by

$$\theta_t(p) = \theta(t, p).$$

The defining properties (*) are equivalent to the **group laws**

$$(*) \quad \theta_t \circ \theta_s = \theta_{t+s}, \quad \theta_0 = \text{Id}_M.$$

As in the case for any group action, each map $\theta_t : M \rightarrow M$ is a homeomorphism, and if the action is smooth, θ_t is a diffeomorphism.

(2) $\forall p \in M$, define $\theta^{(p)} : \mathbb{R} \rightarrow M$ by

$$\theta^{(p)}(t) = \theta(t, p).$$

The image of this curve is just the orbit of p under the group action. Because **any group action partitions the manifold into disjoint orbits**, it follows that M is the disjoint union of the images of these curves.

Definition 10. A family $\{(\theta_t)_{t \in I}\}$ (I open interval with $0 \in I$) of diffeomorphisms from M to M satisfying (*) is called a **local 1-parameter group of diffeomorphisms**.

- In general, a local 1-parameter group need not be extendable to a group since the maximal interval of definition I_q of $\theta^{(p)}(t)$ need not be all of \mathbb{R} .

Example 11. Let $M = \{(x, y) \in \mathbb{R}^2 : x < 0\}$, and let $V = \partial/\partial x$.

— The unique integral curve of V starting at $(-1, 0) \in M$ is $\gamma(t) = (t - 1, 0)$.

— However, in this case, γ cannot be extended past $t = 1$.

Example 12. Let $M = \mathbb{R}^n$ and $W = x^2\partial/\partial x$. The unique integral curve of W starting at $(1, 0)$ is

$$\gamma(t) = \left(\frac{1}{1-t}, 0 \right).$$

This curve cannot be extended past $t = 1$, because it escapes to infinity as $t \nearrow 1$.

Definition. We say that a smooth vector field is **complete** if it generates a global flow, or equivalently if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

- It is often impossible to solve the ODE explicitly, so it is useful to have some general criteria for determining when a vector field is complete.

Theorem 13. Let V be a vector field on M with compact support. Then the corresponding flow is defined for all $q \in M$ and all $t \in \mathbb{R}$, and the local 1-parameter group becomes a group of diffeomorphisms.

Proof. $\forall p \in M$, \exists a nbhd U_p of p and $\varepsilon > 0$ such that $\forall q \in U$, the curve $\theta^{(q)}(t)$ is defined on $(-\varepsilon, \varepsilon)$.

- Let now $\text{supp } X \subset K$, K compact. K can then be covered by finitely many such nbhds, and we choose ε_0 as the smallest such ε .
- Since for $q \notin K$, $V(q) = 0$, $\theta_t(q) = \theta^{(q)}(t)$ is defined on $(-\varepsilon_0, \varepsilon_0) \times M$, and for $|s|, |t| < \varepsilon_0/2$, we have the semigroup property (*).
- Since the interval of existence $(-\varepsilon_0, \varepsilon_0)$ may be chosen uniformly for all q , one may **iteratively extend the flow onto all of \mathbb{R}** . For this purpose, we write $t \in \mathbb{R}$ as

$$t = m \frac{\varepsilon_0}{2} + \rho, \quad \text{with } m \in \mathbb{Z}, 0 \leq \rho < \varepsilon_0/2$$

and set $\theta_t = (\theta_{\varepsilon_0/2})^m \circ \theta_\rho$. $(\theta_t)_{t \in \mathbb{R}}$ then is the desired 1-parameter group. \square

Corollary 14. On a compact smooth manifold, any smooth vector field generates a 1-parameter group of diffeomorphisms, or equivalently, every smooth vector field on M is complete.

Theorem 13* (Escape Lemma). Let V be a smooth vector field on a smooth mfd M . If γ is an integral curve of V whose maximal domain is not all of \mathbb{R} , then the image of γ cannot lie in any compact subset of M .

Proof. Let (a, b) denote the maximal domain of γ , so that $-\infty \leq a < 0 < b \leq +\infty$. Let $p = \gamma(0)$ and let θ denote the flow of V , so $\gamma = \theta^{(p)}$ by the uniqueness of integral curves. Assume $b < \infty$ but $\gamma(a, b)$ lies in a compact set $K \subset M$.

Claim: γ can be extended past b , which contradicts the maximality of (a, b) . (The argument for the case $a > -\infty$ is similar).

- If $\{t_i\}$ is any sequence of times such that $t_i \nearrow b$, then the sequence $\{\gamma(t_i)\}$ lies in K and therefore **has a subsequence converging to a point $q \in M$** .
- \exists a nbhd U of q and a positive number ε such that θ is defined on $(-\varepsilon, \varepsilon) \times U$.
- Pick i so large that $\gamma(t_i) \in U$ and $t_i > b - \varepsilon$, and define $\sigma : (a, t + \varepsilon) \rightarrow M$ by

$$\sigma(t) = \begin{cases} \gamma(t), & a < t < b, \\ \theta_{t-t_i} \circ \theta_{t_i}(p), & t_i - \varepsilon < t < t_i + \varepsilon. \end{cases}$$

Then two definitions agree where they overlap, as $\theta_{t-t_i} \circ \theta_{t_i}(p) = \theta_t(p) = \gamma(t)$ by the group law for θ . Thus σ is an integral curve extending γ , which contradicts the maximality of (a, b) . \square

A Proof of Corollary 14. If M is compact, the escape lemma implies that no integral curve can have a maximal domain that is not all of \mathbb{R} , because the image of any integral curve is contained in the compact set M . \square

Lemma 17. Let X be a smooth vector field on M , $F : M \rightarrow N$ a diffeomorphism. Let the local 1-parameter group generated by X is given by θ_t , and the local group generated by Y is η_t . Then $Y = F_*X$ iff $F \circ \theta_t \circ F^{-1}$ on the domain of θ_t :

$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \theta_t \downarrow & & \downarrow \eta_t \\ M & \xrightarrow{F} & N \end{array}$$

Proof 1. The commutativity of the diagram means that the following holds for all (t, p) in the domain of θ :

$$\eta_t \circ F(p) = F \circ \theta_t(p);$$

that is,

$$(3) \quad \eta^{F(p)}(t) = F \circ \theta^{(p)}(t),$$

for all t in the domain of definition of $\theta^{(p)}$.

(\Rightarrow) Suppose that $Y = F_*X$. If we define γ by $\gamma = F \circ \theta^p$, then

$$\gamma'(t) = (F \circ \theta^{(p)})'(t) = F_*(\theta^{(p)'}(t)) = F_*X_{\theta^{(p)}(t)} = Y_{F \circ \theta^{(p)}(t)} = Y_{\gamma(t)},$$

and hence γ is an integral curve of Y starting at $F \circ \theta^{(p)}(0) = F(p)$.

By **uniqueness** of integral curves, $\gamma(p) = \eta^{F(p)}(t)$ on the interval where $\theta^{(p)}$ is defined. This proves (3).

(\Leftarrow) If (3) holds, then $\forall t \in M$, we have

$$F_*X_p = F_*(\theta^{(p)'}(0)) = (F\theta^{(p)})'(0) = \eta^{F(p)'}(0) = Y_{F(p)}.$$

Proof 2. $F \circ \theta_t \circ F^{-1}$ is a local 1-parameter group, and therefore, by uniqueness of solutions of ODE, it suffices to show the claim near $t = 0$. Now

$$\begin{aligned} \left. \frac{d}{dt}(F \circ \theta_t \circ F^{-1}) \right|_{t=0} &= dF \left(\left. \frac{d}{dt}(\theta_t \circ F^{-1}) \right|_{t=0} \right), \\ &\text{where } dF \text{ is evaluated at } \theta_0 \circ F^{-1}(p) = F^{-1}(p) \\ &= dF_{F^{-1}(p)}X(F^{-1}(p)) = F_*X(p). \quad \square \end{aligned}$$