Lie Derivatives

- The Lie derivative is a method of computing the "directional derivative" of a vector field with respect to another vector field.

- We already know how to make sense of a "directional" derivative of real valued functions on a manifold.
  1. A tangent vector $V \in T_pM$ is by definition an **operator** that acts on a smooth function f to give a number Vf that we interpret as a directional derivative of f at p.
  2. This number can also be interpreted as the ordinary derivative of f along any curve whose initial tangent vector is V.

- What about the directional derivative of a vector field?
- In Euclidean space, we can just differentiate the component functions of the vector field.
- But making sense of directional derivatives of a vector field W on a manifold is not so easy as it is in Euclidean space, and thus cannot be compared directly.

— This problem can be circumvented if we replace the vector $V \in T_pM$ with a **vector field**.
— In this case, we can use the **flow** of the vector field to **push** values of W **back** to p and then differentiate.
— This result is called the **Lie derivative of** W w.r.t. the given vector field.

- In Euclidean space, it makes perfectly good sense to define the directional derivative of a smooth vector field W in the directional of a vector $V \in T_p\mathbb{R}^n$. It is the vector

$$D_V W(p) = \frac{d}{dt}\Big|_{t=0} W_{p+tV} = \lim_{t\to 0} \frac{W_{p+tV} - W_p}{t}. \tag{1}$$

  An easy computation shows that $D_V W$ can be evaluated by applying V to each component of W seperately:

$$D_V W(p) = V W^i(p) \frac{\partial}{\partial x^i}\Big|_p.$$

- Unfortunately, this definition is heavily dependent upon the fact that $\mathbb{R}^n$ is a vector space, so that the tangent vectors W_{p+tV} and W_p can both be viewed as elements of $\mathbb{R}^n$.
- We search for a way to make invariant sense of (1) on a manifold.

— To begin with, we can replace $p + tV$ by a curve $\gamma(t)$ that starts at p and whose initial tangent vector is V.

— But even with this substitution, the difference quotient still makes no sense because $W_{\gamma(t)}$ and $W_{\gamma(0)}$ are elements of **different vector spaces** $T_{\gamma(t)}M$ and $T_{\gamma(0)}M$.

— We got away with it in Euclidean space because **there is a canonical identification of each tangent space with $\mathbb{R}^n$ itself**; but on a manifold there is no such identification.

— Thus there is **no coordinate-independent way** to make sense of the directional derivative of W in the direction of the vector V.

- Now suppose that V itself is a **smooth vector field** instead of a single vector. In this case, we can use the **flow** of V to **push** values of W **back** to p and then differentiate.

Definition. *For any smooth vector fields V and W on a manifold M, let θ be the flow of V, and define a vector $(\mathcal{L}_V W)_p$ at each $p \in W$, called the* **Lie Derivative** *of W with respect to V at p, by*

$$(\mathcal{L}_V W)_p = \frac{d}{dt}\Big|_{t=0} (\theta_{-t})_* W_{\theta_t(p)} = \lim_{t\to 0} \frac{(\theta_{-t})_* W_{\theta_t(p)} - W_p}{t}.$$

provided the derivative exists.
(For small $t \neq 0$, the difference quotient makes sense at least, because θ_t is defined in a nbhd of p, and both $(\theta)_* W_{\theta_t(p)}$ and W_p are elements of T_pM.)

- By (2), we have

$$(\mathcal{L}_V W)_p = -\lim_{t\to 0} \frac{(\theta_t)_* W_{\theta_{-t}(p)} - W_p}{t} = \frac{d}{dt}\Big|_{t=0} (\theta_t)_* W_{\theta_{-t}(p)}.$$

Lemma 1. *If V and W are smooth vector fields on a smooth manifold M, then $(\mathcal{L}_V W)_p$ exists $\forall p \in M$, and the assignment $p \to (\mathcal{L}_V W)_p$ defines a smooth vector field.*

Proof. Let θ be the flow of V. For arbitrary $p \in M$, let $(U, (x^i))$ be a smooth coordinate chart containing p.

— Choose an open interval J_0 containing 0 and an open set $U_0 \subset U$ containing p such that θ maps $J_0 \times U_0$ into U. For $(t, x) \in J_0 \times U_0$, we can write the component functions of θ as $(\theta^1(t, x), \cdots, \theta^n(t, x))$.

Then for any $(t, x) \in J_0 \times U_0$, the matrix of $(\theta_{-t})_* : T_{\theta_t(x)} M \to T_x M$ is

$$\left(\frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j}\right).$$

$$\therefore (\theta_{-t})_* W_{\theta_t(x)} = \frac{\partial \theta^i(-t, \theta(t, x))}{\partial x^j} W^j(\theta(t, x)) \frac{\partial}{\partial x^i}\bigg|_x.$$

— Because θ^i and W^j are smooth, the coefficients $\frac{\partial}{\partial x^i}\big|_x$ depends smoothly on (t, x).

– It follows that $(\mathcal{L}_V W)_x$, which is obtained by taking the derivative of the expression with respect to t and setting $t = 0$ exists for each $x \in U_0$ and depends smoothly on x. □

Theorem 2. *For any smooth vector field X and Y on a smooth manifold M,*

$$\mathcal{L}_X Y = [X, Y].$$

Proof 1. Let $\mathcal{R}(X) \subset M =$the set of regular points of $X = \{p : p \in M,\ V_p \neq 0$.

— Note that $\mathcal{R}(X)$ is open in M by continuity, and its closure is the support of X.

Step 1: Claim: $\mathcal{L}_X Y = [X, Y]$ on $\mathcal{R}(X)$.

— If $p \in \mathcal{R}(X)$, we can choose smooth coordinates (u^i) on a nbhd of X in which X has the coordinate representation $X = \partial/\partial u^1$.

— In these coordinates, the flow of X is

$$\theta_t(u) = (u^1 + t, u^2, \cdots, u^n).$$

Consequently, for any $u \in U$,

$$(\theta_{-t})_* Y_{\theta_t(u)} = (\theta_{-t})_* \left(Y^j(u^1 + t, u^2, \cdots, u^n)\right) \frac{\partial}{\partial u^j}\bigg|_u = \frac{\partial Y^j}{\partial u^1}(u^1, \cdots, u^n) \frac{\partial}{\partial u^j}\bigg|_u.$$

Using the definition of the Lie derivative, we obtain

$$\begin{aligned}(\mathcal{L}_X Y)_u =& \frac{d}{dt}\bigg|_{t=0} Y^j(u^1 + t, u^2, \cdots, u^n) \frac{\partial}{\partial u^j}\bigg|_u \\ =& \frac{\partial Y^j}{\partial u^1}(u^1, \cdots, u^n) \frac{\partial}{\partial u^j}\bigg|_u = [X, Y]_u.\end{aligned}$$

Step 2: Claim: $\mathcal{L}_X Y = [X, Y]$ on $\operatorname{supp} X$.

Because $\operatorname{supp} X$ is the closure of $\mathcal{R}(X)$, this follows from Step 1 by continuity.

Step 3: Claim: $\mathcal{L}_X Y = [X, Y]$ on $M \setminus \operatorname{supp} X$.

— If $p \in M \setminus \operatorname{supp} X$, then $X \equiv 0$ on a nbhd of p.

– On the one hand, this implies θ_t is the identity map in a nbhd of p for all t, so $(\theta_{-t})_* Y_{\theta_t(p)} = Y_p$, which implies $(\mathcal{L}_V W)_p = 0$.

– On the other hand, $[X, Y]_p = 0$. □

Proof 2. Let $Y = Y^i \frac{\partial}{\partial x^i}$. We have

$$\begin{aligned}
\mathcal{L}_X Y =& \frac{d}{dt}(\theta_{-t})_*(Y^i \frac{\partial}{\partial x^i})\Big|_{t=0} \\
=& \frac{d}{dt}(Y^i(\theta_t))\frac{\partial \theta^j_{-t}}{\partial x^j}\frac{\partial}{\partial x^j}\Big|_{t=0} \\
=& \frac{\partial Y^i}{\partial x^k} X^k \delta^j_i \frac{\partial}{\partial x^j} + Y^i(-\frac{\partial X^j}{\partial x^i})\frac{\partial}{\partial x^j}, \quad (\because \theta_0 = \mathrm{id}, \quad \frac{d}{dt}\theta_{-t}\Big|_{t=0} = -X), \\
=& (X^k \frac{\partial Y^j}{\partial x^k} - Y^k \frac{\partial X^j}{\partial x^k})\frac{\partial}{\partial x^j} = [X, Y]. \qquad \square
\end{aligned}$$

Proof 3. Let $f \in C^\infty(M)$. Then

$$\begin{aligned}
X_p Y f =& \theta^{(p)\prime}(0) Y f = \frac{d}{dt}\Big|_{t=0} Y f(\theta^{(p)}(t)) \\
=& \lim_{t\to 0} \frac{Y f(\theta^{(p)}(t)) - Y f(p)}{t} = \lim_{t\to 0} \frac{Y f(\theta_t(p)) - Y f(p)}{t}
\end{aligned}$$

$$\begin{aligned}
Y_{\theta_{t^*}(p)} X f =& Y_{\theta_{t^*}(p)}(\theta^{(p)\prime}(\tilde{t}) f) = Y_{\theta_t(p)}\left(\lim_{t\to 0} \frac{(f(\theta^{(p)}(\tilde{t})) - f(\theta^{(p)}(\tilde{t} - t))}{t}\right) \\
=& Y_{\theta_{t^*}(p)}\left[\left(\lim_{t\to 0} \frac{f(\theta_{\tilde{t}}(p)) - f(\theta_{\tilde{t}-t}(p))}{t}\right)\right] \\
=& \left[Y_{\theta_{t^*}(p)}\left(\lim_{t\to 0} \frac{f - f\circ\theta_{-t}}{t}\right)\circ\theta_{\tilde{t}-t^*}(p)\right]\circ\theta_{t^*}(p) \\
=& \left[Y_{\theta_{t^*}(p)}\left(\lim_{t\to 0} \frac{f - f\circ\theta_{-t}}{t}\right)\right]\circ\theta_{t^*}(p)
\end{aligned}$$

since $|t^* - \tilde{t}|$ can be arbitrarily small and Y is smooth. On the other hand, we have

$$\begin{aligned}
&\frac{(\theta_{-t})_* Y_{\theta_t(p)} f - Y_p f}{t} = \frac{Y_{\theta_{t^*}(p)}(f\circ\theta_{-t}(p)) - Y f(p)}{t} \\
&= \frac{(Y_{\theta_t(p)}(f\circ\theta_{-t}(p)))\circ\theta_t(p) - Y f(\theta_t(p))}{t} + \frac{Y f(\theta_t(p)) - Y f(p)}{t} \\
&= \left[Y_{\theta_t(p)}\left(\frac{f\circ\theta_{-t} - f}{t}\right)\right]\circ\theta_t(p) + \frac{Y f(\theta_t(p)) - Y f(p)}{t}.
\end{aligned}$$

Hence

$$\lim_{t\to 0} \frac{(\theta_{-t})_* Y_{\theta_t(p)} f - Y_p f}{t} = -\lim_{t\to 0} Y_{\theta_t(p)} X f + X_p Y f = X_p Y f - Y_p X f. \qquad \square$$

- Theorem 2 gives a geometric interpretation of the Lie bracket of two vector fields: **It is the directional derivative of the second vector field along the flow of the first**.
- A number of nonobvious properties of the Lie derivative now follow immediately from things we already know about Lie brackets.

Corollary 3. *Suppose V, W, $X \in \mathcal{T}(M)$ and $f \in C^\infty(M)$.*

(a) $\mathcal{L}_V W = -\mathcal{L}_W V$.

(b) $\mathcal{L}_V [W, X] = [\mathcal{L}_V W, X] + [W, \mathcal{L}_V X]$.

(c) $\mathcal{L}_{[V,W]} X = \mathcal{L}_V \mathcal{L}_W X - \mathcal{L}_W \mathcal{L}_V X$.

(d) $\mathcal{L}_V (fW) = (Vf)W + f\mathcal{L}_V W$.

(e) If $F : M \to N$ is a diffeomophism, then $F_*(\mathcal{L}_V W) = \mathcal{L}_{F_* V} F_* W$.