Lie Derivatives of Tensor Fields

• Any Lie derivative on vector fields automatically induces Lie derivatives on all
tensor bundles over $M$, and thus gives us a way to compute Lie derivatives of all
tensor fields.
• We want $\mathcal{L}$ obeys the following product rule w.r.t. the natural pairing between
a covector field $\omega$ and a vector field $Y$:

$$\mathcal{L}_X(\omega(Y)) = \mathcal{L}_X\omega(Y) + \omega(\mathcal{L}_X Y).$$

Thus, it seems reasonable to define $\mathcal{L}_X\omega$ as

$$\mathcal{L}_X\omega(Y) = \mathcal{L}_X(\omega(Y)) - \omega(\mathcal{L}_X Y).$$

Lemma 1. Let $\mathcal{L}$ be a Lie derivative on $\Gamma(TM)$. There is a unique Lie derivative
in each tensor bundle $T^k_m M$, also denoted by $\mathcal{L}$, such that the following conditions
are satisfied.

(a) $\mathcal{L}_X f = Xf$, for all smooth real-valued functions $f$;
(b) $\mathcal{L}_X( f \sigma) = (\mathcal{L}_X f) \sigma + f \mathcal{L}_X \sigma$, for all smooth tensor fields $\sigma$, $\tau$;
(c) $\mathcal{L}_X(\sigma \otimes \tau) = (\mathcal{L}_X \sigma) \otimes \tau + \sigma \otimes \mathcal{L}_X \tau$, for smooth tensor fields $\sigma$, $\tau$;
(d) If $Y_1, \ldots, Y_k$ are smooth vector fields and $\sigma$ is a smooth $k$-tensor field, then

$$\mathcal{L}_X(\sigma(Y_1, \ldots, Y_k)) = (\mathcal{L}_X \sigma)(Y_1, \ldots, Y_k)
+ \cdots + \sigma(Y_1, \ldots, \mathcal{L}_X Y_k).$$

Corollary 2. If $X$ is a smooth vector field and $\sigma$ is a smooth covariant $k$-tensor
field, then for any smooth vector fields $Y_1, \ldots, Y_k$,

$$(\mathcal{L}_X \sigma)(Y_1, \ldots, Y_k)) = X(\sigma(Y_1, \ldots, Y_k)) - \sigma([X, Y_1, Y_2, \ldots, Y_k]) - \cdots
- \sigma(Y_1, \ldots, Y_{k-1}, [X, Y_k]).$$

$\Box$ For any $p \in M$, it is sufficiently close to zero, $\theta_t$ is a diffeomorphism from a nbhd
of $p$ to a nbhd of $\theta_t(p)$, so $\theta_t^*\omega$ pulls back tensors at $\theta_t(p)$ to ones at $p$.

Lemma 2. Let $X$ be a smooth vector field on smooth manifold $M$, and let $\theta$ be
its flow. Given a smooth covariant tensor field $\tau$ on $M$, the Lie derivative of $\tau
$ with respect to $X$

$$(\mathcal{L}_X \tau)_p = \frac{d}{dt} \bigg|_{t=0} (\theta^*_t \tau)_p = \lim_{t \to 0} \frac{\theta^*_t(\tau_{\theta_t(p)} - \tau_p)}{t}.$$ 

Proof. We have

$$\frac{\theta^*_t \tau_{\theta_t(p)}(Y_p) - \tau_p(Y_p)}{t} = \frac{\tau_{\theta_t(p)}((\theta_t(p))_* Y_p) - \tau_{\theta_t(p)}(Y_p)}{t}$$

$$= \frac{\tau_{\theta_t(p)}((\theta_t(p))_* Y_p - Y_{\theta_t(p)}) + (\tau_{\theta_t(p)}(Y_{\theta_t(p)} - \tau_p(Y_p))}{t}$$

$$\to -(\tau(\mathcal{L}_{\theta_t(p)}) Y_p) + (\theta_t(p)')(0)(\tau(Y))_p, \text{ as } t \to 0,$$

$$= -(\tau(\mathcal{L}_X Y))_p + X(\tau(Y))_p. \quad \Box$$
Covariant Derivatives of Tensor Fields

- By definition, a connection on $M$ is a way to compute covariant derivatives of vector fields.
- In fact, any connection automatically induces connections on all tensor bundles over $M$, and thus gives us a way to compute covariant derivatives of all tensor fields.
- We want $D$ obeys the following product rule w.r.t. the natural pairing between a covector field $\omega$ and a vector field $Y$:

$$D_X(\omega(Y)) = D_X\omega(Y) + \omega(D_X Y).$$

Thus, it seems reasonable to define $D_X\omega$ as

$$D_X\omega(Y) = D_X(\omega(Y)) - \omega(D_X Y).$$

**Lemma 1.** Let $D$ be a connection on $M$. There is a unique connection in each tensor bundle $T^k_\ell M$, also denoted by $D$, such that the following conditions are satisfied.

(a) On $TM$, $D$ agrees with given connection.
(b) On $T^0M = C^\infty(M)$, $D$ is the ordinary differentiation of functions: $D_X f = X f$.
(c) $D$ obeys the following product rule w.r.t. tensor products:

$$D_X(F \otimes G) = (D_X F) \otimes G + F \otimes (D_X G).$$

This connection satisfies the following properties:

(i) $D$ obeys the following product rule w.r.t. the natural pairing between a covector field $\omega$ and a vector field $Y$:

$$D_X(\omega(Y)) = D_X\omega(Y) + \omega(D_X Y).$$

(ii) For any $F \in \Gamma(T^k_\ell M)$, vector fields $Y_i$ and 1-form $\omega^j$,

$$(D_X F)(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, Y_k) = X(F(\omega^1, \ldots, \omega^\ell, \omega^1, \ldots, Y_k)) - \sum_{j=1}^{\ell} F(\omega^1, \ldots, D_X\omega^j, \ldots, \omega^\ell, Y_1, \ldots, Y_k) - \sum_{i=1}^{k} F(\omega^1, \ldots, \omega^\ell, Y_1, \ldots, D_X Y_i, \ldots, Y_k).$$
If $S$ is a $(r_0^0)$ or $(r_1^1)$ tensor field, then we can define a **covariant derivative** $DS$ that we interpret as a $(r_0^{r_0^1})$ or $(r_1^{r_1^1})$-tensor field.

The main idea is to make sure that the **product rule** holds. Thus if $S$ is a $(r_1^1)$ tensor field, we want to have

$$D_X(S(Y)) = (DXS)(Y) + S(DXY).$$

Thus, it seems reasonable to define $D$ as

$$DS(X, Y) = (DXS)(Y) - S(DXY).$$

More generally, we define

$$DS(X, Y_1, \cdots, Y_r) = (DXS)(Y_1, \cdots, Y_r)
= DX(S(Y_1, \cdots, Y_r)) - \sum_{i=1}^r S(Y_1, \cdots, DXY_i, \cdots, Y_r),$$

where $DX$ is interpreted as the directional derivative when applied to a function, while we use it as covariant differentiation on vector fields.

**Lemma 2.** If $D$ is a linear connection on $M$, and $S \in \Gamma(T^k_M)$, then the map

$$DS(\omega^1, \cdots, \omega^\ell, Y_1, \cdots, Y_k, X) = DXS(\omega^1, \cdots, \omega^\ell, Y_1, \cdots, Y_k),$$

defines a $(k+1^\ell)$-tensor field.

**Definition.** The tensor field $DS$ is called the **total covariant derivative**.

**Lemma 3.** Let $D$ be a linear connection. The components of the total covariant derivative of a $(k_\ell)$-tensor field $F$ with respect to a coordinate system are given by

$$F_{j_1\cdots j_k}^{j_1\cdots j_\ell} = \partial_m F_{j_1\cdots j_k}^{j_1\cdots j_\ell} + \sum_{s=1}^\ell \Gamma_{mp}^{j_s} F_{i_1\cdots i_k}^{j_1\cdots j_s} - \sum_{s=1}^k \Gamma_{mp}^{j_1\cdots j_s} F_{i_1\cdots i_k}^{j_1\cdots j_\ell}.$$ 

**Definition.** A tensor field $S$ is said to be **parallel** if $DS = 0$.

On a Riemannian manifold $(M, g)$, we always have that $Dg = 0$, since $(Dg)(X, Y_1, Y_2) = DX(g(Y_1, Y_2)) - g(DXY_1, Y_2) - g(Y_1, DXY_2) = 0$. 

• Let $f$ be a smooth function on $M$. Then $Df \in T^* M$ is just the 1-form $df$, because both tensors have the same action on tensors:

$$Df(X) = D_X f = X f = df(X).$$

• The $(20)$-tensor $D^2 f = D(Df)$ is called the covariant Hessian of $f$. The covariant Hessian is symmetric:

$$D^2 f(X, Y) = D_X (df(Y)) - df(D_X Y)
= X(Y f) - df(D_X Y)
= X( Y f) - df(D_Y X) - [X, Y] f
= Y(X f) - df(D_Y X)
= D^2 f(Y, X).$$

**Definition.** For a $(r)$-tensor field $S$ we define the second covariant derivative $D^2 S$ as the $(r+2)$-tensor field

$$(D_{X_1, X_2}^2 S)(Y_1, \cdots, Y_r) = (D_{X_1} (DS))(X_2, Y_1, \cdots, Y_r)
= D_{X_1} (DS(X_2, Y_1, \cdots, Y_r)) - (DS)(D_{X_1} X_2, Y_1, \cdots, Y_r)
= D_{X_1} D_{X_2} S(Y_1, \cdots, Y_r) - (DS)(D_{X_1} X_2, Y_1, \cdots, Y_r)
= (D_{X_1} D_{X_2} S)(Y_1, \cdots, Y_r) - (D_{D_{X_1}, x_2} S)(Y_1, \cdots, Y_r).$$

• With this we obtain the $(20)$ version of the Hessian of a function defined as

$$D^2_{X, Y} f = D_X D_Y f - D_{D_X Y} f
= D_X (df(Y)) - df(D_X Y)
= D_X df(Y)
= D^2 f(X, Y),$$

which is symmetric in $X$ and $Y$. 