The Tangent-Cotangent Isomorphism

- A very important feature of any Riemannian metric is that **it provides a natural isomorphism between the tangent and cotangent bundles**.

- Let (M, g) be a Riemannian manifold. For each point $p \in M$, there is a positive-definite inner product
$$g_p : T_pM \times T_pM \to \mathbb{R}.$$
By setting
$$\widetilde{g}_p(X)(Y) = g_p(X, Y).$$
we obtain a linear map
$$\widetilde{g}_p : T_pM \to T_p^*M.$$

- **Claim:** $\widetilde{g}_p$ **is an isomorphism;** i.e. by means of the metric g_p we can identify the tangent space T_pM and the cotangent space T_p^*M. Indeed,

  (1) $\widetilde{g}_p$ is injective, since $\widetilde{g}_p(X) = 0 \Rightarrow \widetilde{g}_p(X)(X) = 0 \overset{\text{i.e.}}{\Rightarrow} g(X, X) = 0 \Rightarrow X = 0$.
  (2) On the other hand, we have $\dim T_pM = \dim T_p^*M$, and hence $\widetilde{g}_p$ is bijective.

— We use the same symbol for both the pointwise bundle map $\widetilde{g} : TM \to T^*M$ and the linear map on sections $\widetilde{g} : \Gamma(TM) \to \Gamma(T^*M)$; namely, we have the following.

Definition. *Given a Riemannian metric g on a manifold M, define a bundle map*
$$\widetilde{g} : TM \to T^*M$$
as follows: $\forall p \in M$ and $\forall X_P \in T_pM$, we let $\widetilde{g}(X_p) \in T_p^*M$ be the covector defined by
$$\widetilde{g}(X_p)(Y_p) = g_p(X_p, Y_p), \quad \forall Y_p \in T_pM.$$

- To see that this is a **smooth** bundle map, it is easiest to consider its action on smooth vector fields:
$$\widetilde{g}(X)(Y) = g(X, Y), \quad \forall X, Y \in \mathcal{T}(M).$$

— Because $\widetilde{g}(X)(Y)$ is linear over $C^\infty(M)$ as a function of Y, it follows that $\widetilde{g}(X)$ is a smooth covector field;
and because $\widetilde{g}(X)(Y)$ is linear over $C^\infty(M)$ as a function of X, this defines $\widetilde{g}$ as a smooth bundle map.

- If X and Y are smooth vector fields, in smooth coordinates we can write
$$\widetilde{g}(X)(Y) = g_{ij}X^iY^j,$$
which implies that the covector field $\widetilde{g}(X)$ has the coordinate expression
$$\widetilde{g}(X) = g_{ij}X^i dy^j.$$

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In other words, $\widetilde{g}$ is the bundle map whose matrix w.r.t. coordinate frames for TM and T^*M is the same as the matrix of g itself.

- It is customary to denote the component of the covector field $\widetilde{g}(X)$ by

$$X_j = g_{ij} X^i,$$

so that

$$\widetilde{g}(X) = X_j dy^j.$$

— Because of this, one says that $\widetilde{g}(X)$ is obtained from X by **lowering an index**.
— The notation $X^\flat$ is frequently used for $\widetilde{g}(X)$.

- The matrix of the inverse map $\widetilde{g}^{-1} : T_p^*M \to T_pM$ is thus the inverse of (g_{ij}). (Because (g_{ij}) is the matrix of the isomorphism $\widetilde{g}$, it is invertible at each point.) Let (g^{ij}) denote the matrix-valued function whose value at $p \in M$ is the inverse of the matrix $(g_{ij}(p))$, so that $g^{ij}g_{jk} = g_{kj}g^{ij} = \delta^i_k$.

⊙ For a covector field $\omega \in \mathcal{T}^*M$, $\widetilde{g}^{-1}(\omega)$ has the coordinate representation

$$\widetilde{g}^{-1}(\omega) = \omega^i \frac{\partial}{\partial x^i}, \quad \text{where } \omega^i = g^{ij}\omega_j.$$

— We use tha symbol $\omega^\sharp$ for $\widetilde{g}^{-1}(\omega)$,
and say that $\omega^\sharp$ is obtained from ω by **raising the index**.

- $\omega^\sharp$ is a vector, which we visualize as a (sharp) arrow;
  $X^\flat$ is a covector, which we visaulize by means of its (flat) level sets.

- The most important use of the sharp operation is to reinstate the **gradient** as a vector field on a Riemannian manifold.

Definition. *For any smooth real-valued function f on a Riemannian manifold (M, g), we define a vector field called the* **gradient** *of f by*

$$\operatorname{grad} f = (df)^\sharp = \widetilde{g}^{-1}(df).$$

- Unravelling the definition, we see that $\forall X \in \mathcal{T}(M)$, it satisfies

$$\langle \operatorname{grad} f, X\rangle_g = \widetilde{g}(\operatorname{grad} f)(X) = df(X) = Xf.$$

Thus $\operatorname{grad} f$ is the unique vector field that satisfies

$$\langle \operatorname{grad} f, X\rangle_g = Xf, \quad \text{for every vector field } X,$$

or equivalently,

$$\langle \operatorname{grad} f, \cdot\rangle_g = df.$$

- In smooth coordinates, $\operatorname{grad} f$ has the expression

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

— In particular, this shows that grad f is smooth.

Example. On $\mathbb{R}^n$ with the Euclidean metric, this reduces to

$$\operatorname{grad} f = \delta^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^i}.$$

Example. The matrix of $\overline{g}$ in polar coordinates is

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

so its inverse matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Inserting this into the formula for the gradient, we obtain

$$\operatorname{grad} f = \frac{\partial f}{\partial r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta}.$$

$$\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

- In the general case, suppose grad f has no zero point in M. Then we have the notion of "level hypersurface", namely, a hypersurface on which f is constant and grad f is perpendicular to each level hypersurface.

- For any two 1-forms ω and η, we have their inner product $\langle \omega_p, \eta_p \rangle$ at each point p, and so we have a function $\langle \omega, \eta \rangle$ on M.
- We shall **generalize this fact to the case of k-forms**.

⊙ Indeed, let V be a vector space.
(i) Using a positive-definite inner product given on V,
we induce an isomorphism $V \cong V^*$ and hence an inner producct in V^* as well.
(ii) Let $k \geq 1$. For two elements of the form $\alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta_1 \wedge \cdots \wedge \beta_k$ (α_i, $\beta_j \in V^*$), we define the value of their inner product to be

$$(\alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \cdots \wedge \beta_k) = \det(\langle \alpha_i, \beta_j \rangle).$$

That this value is **independent of the way the two elements are represented** follows from the properties of exterior product and determinant.
— We now extend the inner product so defined to the whole space $\Lambda^k V^*$ by linearity.
– If $e_1, \cdots, e_n$ is an orthonormal basis of V and $\theta_1, \cdots, \theta_n$ the dual basis, then all the elements of the form

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}, \quad 1 \leq i_1 < \cdots < i_k \leq n,$$

form an orthonormal basis of $\Lambda^k V^*$.
– In this way, for any two k-forms ω and η on M, we have the inner product $\langle \omega_p, \eta_p \rangle$ at each point p, and so we have a function $\langle \omega, \eta \rangle$ on M.
(iii) Note that in the special case $k = 0$, we define the inner product between functions f and g at p.

⊙ We also define the inner product between two differential forms of different degree to be 0.

Example 6. For two 2-forms on $\mathbb{R}^3$

$$\begin{aligned}\omega =& a dx_1 \wedge dx_2 + b dx_2 \wedge dx_3 + c dx_3 \wedge dx_1 \\ \eta =& e dx_1 \wedge dx_2 + f dx_2 \wedge dx_3 + g dx_3 \wedge dx_1,\end{aligned}$$

we have $\langle \omega, \eta \rangle = ae + bf + cg$.