Stokes's Theorem

- Stoke's theorem is a far-reaching generalization of the fundamental theorem of calculus and of the classical theorems of vector calculus.

Stokes's Theorem. *Let M be a smooth, oriented n-dimensional manifold with boundary. Let ω be a compactly supported smooth $(n-1)$-form on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{1}$$

Remarks. (1) ∂M is understood to have induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\omega\big|_{\partial M}$.

(2) If $\partial M = \emptyset$, then the right-hand side is to be interpreted as zero.

(3) If $\dim M = 1$, the right-hand side is a finite sum.

Proof. **(I)** Begin by considering a special case: **Suppose** $M = \mathbb{R}^n_+$.
Then the fact that ω has compact support means that $\exists$a number $R > 0$ such that $\operatorname{supp}\omega$ is contained in the rectangle

$$A = [-R, R] \times \cdots \times [-R, R] \times [0, R].$$

We can write ω in standard coordinates as

$$\begin{aligned}
\omega &= \sum_{i=1}^n \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \\
\therefore d\omega &= \sum_{i=1}^n d\omega_i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n \frac{\partial \omega_i}{\partial x^j}\, dx^j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i}\, dx^1 \wedge \cdots \wedge dx^n. \\
\therefore \int_{\mathbb{R}^n_+} d\omega &= \sum_{i=1}^n (-1)^{i-1} \int_A \frac{\partial \omega_i}{\partial x^i}\, dx^1 \wedge \cdots \wedge dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^1 \cdots dx^n
\end{aligned}$$

By the fundametal theorem of calculus, the terms with $i \neq n$ reduce to

$$\begin{aligned}
&\sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^1 \cdots dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial x^i}(x)\, dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= \sum_{i=1}^n (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \left[\omega_i(x)\right]_{x^i=-R}^{x^i=R} dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n \\
&= 0, \quad \text{since we have chosen } R \text{ large enough that } \omega = 0 \text{ when } x^i = \pm R.
\end{aligned}$$

The only term that might not be zero is the one for which $i = n$, which is

$$\int_{\mathbb{R}^n_+} d\omega = \sum_{i=1}^{n-1} (-1)^{n-1} \int_{-R}^{R} \int_{-R}^{R} \cdots \int_{0}^{R} \frac{\partial \omega_n}{\partial x^n}(x)\, dx^n dx^1 \cdots dx^{n-1}$$

$$= \sum_{i=1}^{n} (-1)^{n-1} \int_{-R}^{R} \cdots \int_{-R}^{R} [\omega_n(x)]_{x^n=0}^{x^n=R}\, dx^i dx^1 \cdots \widehat{dx^i} \cdots dx^n$$

$$(2) \qquad = \sum_{i=1}^{n} (-1)^{n} \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x^1, \cdots, x^{n-1}, 0)\, dx^1 \cdots \widehat{dx^i} \cdots dx^n$$

since we have chosen R large enough that $\omega = 0$ when $x^n = R$.

— To compare this term to the other side of (1), we compute as follows:

$$\int_{\partial \mathbb{R}^n_+} \omega = \sum_i \int_{A \cap \partial \mathbb{R}^n_+} \omega_i(x^1, \cdots, x^{n-1}, 0) dx^1 \cdots \widehat{dx^i} \cdots dx^n.$$

Since x^n vanishes on $\partial \mathbb{R}^n_+$, we have $dx^n \Big|_{\partial \mathbb{R}^n_+} = 0$.

Thus the only term above that is nonzero is the one for which $i = n$, namely,

$$\int_{\partial \mathbb{R}^n_+} \omega = \int_{A \cap \partial \mathbb{R}^n_+} \omega_n(x^1, \cdots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

– Taking into account that the coordinates $(x^1, \cdots, x^{n-1})$ are positive oriented for $\partial \mathbb{R}^n_+$ when n is even and negatively oriented when n is odd, this becomes

$$\int_{\partial \mathbb{R}^n_+} \omega = (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \omega_n(x^1, \cdots, x^{n-1}, 0)\, dx^1 \cdots dx^{n-1},$$

which is equal to (2).

(II) Next let M be an **arbitrary** smooth manifold with boundary, but consider an n-form ω that is **compactly supported in the domain of a single smooth chart** (U, φ)**.**

Assume w.l.o.g. that φ is an oriented chart, the definition yields

$$\int_M d\omega = \int_{\mathbb{R}^n_+} (\varphi^{-1})^* d\omega = \int_{\mathbb{R}^n_+} d((\varphi^{-1})^* \omega),$$

since $(\varphi^{-1})^* d\omega$ is compactly supported on $\mathbb{R}^n_+$.

– By the result in **(I)**, we have

$$(3) \qquad \int_{\mathbb{R}^n_+} d((\varphi^{-1})^* \omega) = \int_{\partial \mathbb{R}^n_+} (\varphi^{-1})^* \omega,$$

where $\partial \mathbb{R}^n_+$ is given in the induced orientation.

– Since φ_* takes outward-pointing vectors on ∂M to outward-pointing vectors on $\mathbb{R}^n_+$, it follows that $\varphi\Big|_{U\cap\partial M}$ is an orientation-preserving diffeomorphism onto $\varphi(U)\cap\partial\mathbb{R}^n_+$, and thus the right-hand side of (3)

$$\int_{\partial\mathbb{R}^n_+}(\varphi^{-1})^*\omega = \int_{\partial M}\omega.$$

This proves the theorem in this case.

(III) Finally, let ω be an arbitrarily compactly supported smooth $(n-1)$-form.
Choose a cover of $\operatorname{supp}\omega$ by finitely many oriented smooth charts $\{(U_i,\varphi_i)\}$,
and choose a subordinate smooth partition of unity $\{\psi_i\}$,
we can apply the preceeding argument to $\psi_i\omega$ for each i and obtain

$$\begin{aligned}\int_{\partial M}\omega &= \sum_i\int_{\partial M}\psi_i\omega = \sum_i\int_M d(\psi_i\omega) = \sum_i\int_M d\psi_i\wedge\omega+\psi_i d\omega\\ &= \int_M d(\sum_i\psi_i)\wedge\omega + \int_M(\sum_i\psi_i)d\omega = 0+\int_M d\omega,\end{aligned}$$

because $\sum_i\psi_i\equiv 1$. □

Example. Let N be a smooth manifold and suppose that

$$\gamma : [a,b]\to N$$

is a smooth embedding, so that $M=\gamma([a,b])$ is an embedded 1-submanifold with boundary in N.
If we give M an orientation such that γ is orientation-preserving, then for any smooth function $f\in C^\infty(N)$, Stokes's theorem says that

$$\int_\gamma df = \int_{[a,b]}\gamma^* df = \int_M df = \int_{\partial M} f = f(\gamma(b)) - f(\gamma(a)),$$

which reduces to the fundamental theorem for line integrals in this case.
In particular, when $\gamma : [a,b]\to\mathbb{R}$ is the inclusion map, Stokes's theorem is the ordinary fundamental theorem of calculus.

Corollary 2. *Suppose M is a compact smooth manifold without boundary. If ω is an exact form over M, then the integral of ω over ∂M is zero:*

$$\int_M d\omega = 0 \quad \text{if } \partial M = \emptyset.$$

Corollary 3. *Suppose M is a compact smooth manifold with boundary. If ω is a closed form on M, then the integral of ω over ∂M is zero:*

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0 \text{ on } M.$$

Corollary 4. *Suppose M is a smooth manifold, $S \subset M$ is a compact k-dimensional submanifold without boundary.*
Suppose ω is a closed k-form on M such that

$$\int_S \omega \neq 0.$$

Then ω is **not** *exact and S is* **not** *the boundary of a smooth, compact submanifold with boundary in M.*

Example. The closed 1-form

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

has nonzero integral over $\mathbb{S}^1$. Hence ω is not exact on $\mathbb{R}^2 \setminus \{0\}$, and $\mathbb{S}^1$ is not the boundary of a smooth, compact, 2-dimensional submanifold with boundary in $\mathbb{R}^2 \setminus \{0\}$.

Green's Theorem. *Suppose D is a smooth, compact domain in $\mathbb{R}^2$, and P, Q are smooth real-valued functions on D. THen*

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial D} Pdx + Qdy.$$

Proof. This is Stokes's theorem applied to the 1-form $P\,dx + Q\,dy$. □