Harmonic 1-Forms (The Bochner Technique)

- Suppose that $\omega$ is a harmonic 1-form on $(M, g)$. We shall consider
  \[ f = \frac{1}{2} g(\omega, \omega). \]

  One of the technical problems is that we do not have a really good feeling for what $g(\omega, \omega)$ is.
  - If $X$ is the vector field dual to $\omega$, i.e. $\omega(v) = g(X, v)$ for all $v \in \Gamma(TM)$, then,
    \[ f = \frac{1}{2} g(\omega, \omega) = \frac{1}{2} g(X, X) = \frac{1}{2} \omega(X). \]

**Proposition 1.** If $X$ is a vector field on $(M, g)$ and $\omega(v) = g(X, v)$ is the dual 1-form, then
\[ \text{div} X = -\delta \omega. \]

**Proposition 2.** If $X$ is a vector field on $(M, g)$ and $\omega(v) = g(X, v)$ is the dual 1-form, then $v \mapsto D_v X$ is symmetric iff $d\omega = 0$.

**Proof.** Recall that
\[ d\omega(V, W) = D_V \omega(W) - D_W \omega(V) - \omega([V, W]). \]

Using that $\omega(Z) = g(X, Z)$, we then obtain
\[
\begin{align*}
  d\omega(V, W) &= D_V g(X, W) - D_W g(X, V) - g() - g(X, D_V W) + g(X, D_W V) \\
  &= [g(D_V X, W) + g(X, D_W V)] - [g(D_W X, V) + g(X, D_V V)] \\
  &\quad - g(X, D_V W) + g(X, D_W V) \\
  &= g(D_V X, W) - g(D_W X, V). \quad \Box
\end{align*}
\]

**Corollary.** If $\omega$ is harmonic and $X$ is the dual vector field, we have that $\text{div} X = 0$ and $DX$ is a symmetric (1,1)-tensor.

**Proposition 3.** Let $X$ be a vector field so that $DX$ is symmetric. If we define a symmetric (1,1)-tensor by $S(v) = D_v X$, then
\[
\begin{align*}
  (a) \ rac{1}{2} \nabla |X|^2 &= DX, \\
  (b) \ \frac{1}{2} D_V \nabla |X|^2 &= R(X, V)X + (D_X S)(V) + S^2(V). \\
  (c) \ \frac{1}{2} \Delta |X|^2 &= \Delta f = \text{Ric}(X, X) + g(X, \nabla \text{div} X) + |DX|^2.
\end{align*}
\]

**Proof.** (a) Let $f = \frac{1}{2} |X|^2$. Observe that, for all $V \in \Gamma(TM)$,
\[
\begin{align*}
  g(\nabla f, V) = df(V) &= V f = D_V \frac{1}{2} g(X, X) \\
  &= g(D_V X, X) \\
  &= g(D_X X, V), \quad \text{since } DX \text{ is symmetric.}
\end{align*}
\]
(b) Note that

$$\frac{1}{2} D_V \nabla |X|^2 = D_V D_X X, \text{ by (a)}$$

$$= R(X, V) X + D_X D_V X + D_{[V, X]} X$$

$$= R(X, V) X + (D_X D_V X - D_{D_V X} X) + D_{D_V X} X$$

$$= R(X, V) X + D_{[V, X]} X$$

$$= R(X, V) X + (D_X S)(V) + S^2(V).$$

(C) We have to compute the trace of each one of the terms in (b).

– Clearly, $\text{tr}(V \mapsto R(X, V) X) = \text{Ric}(X, X)$.

– Next, since the tensor $v \mapsto D_V X = S(v)$ is assumed to be symmetric, we have

$$|DX|^2 = \text{tr}(D_X \circ DX) = \text{tr}(S^2).$$

– For the last tensor, we shall prove something slightly more general. Namely,

**Lemma 4.** For any $(1,1)$-tensor $S$ and vector field $X$, we have

$$\text{tr}(D_X S) = D_X \text{tr}S.$$

From this we obtain

$$D_X \text{tr}S = D_X \text{div} X = g(X, \nabla \text{div} X).$$

To prove Lemma 4, we assume first the truth of Lemma 5 below and then calculate

$$\text{tr}(D_X S) = \sum g((D_X S)(E_i), E_i)$$

$$= \sum g(D_X (S(E_i)), E_i) - \sum g(S(D_X E_i), E_i)$$

$$= D_X g(S(E_i), E_i) - \sum g(S(E_i), D_X E_i) - \sum g(S(D_X E_i), E_i)$$

$$= D_X g(S(E_i), E_i), \text{ by Lemma 5},$$

$$= D_X \text{tr}S.$$

**Lemma 5.** For any $(1,1)$-tensor $S$ and vector field $X$, we have

$$\sum g(S(E_i), D_X E_i) + \sum g(S(D_X E_i), E_i) = 0,$$

for any orthonormal frame $\{E_i\}$.

**Proof of Lemma.** We observe that, since $\{E_i\}$ is orthonormal, $g(E_i, E_j) \equiv 1$, and hence

$$0 = D_X g(E_i, E_j)$$

$$= g(D_X E_i, E_j) + g(E_i, D_X E_j).$$
Now if $S^*$ denote the adjoint to $S$, we have

\[
\sum g(S(E_i), D_X E_i) + \sum g(S(D_X E_i), E_i) = \sum g(S(E_i), D_X E_i) + \sum g(D_X E_i, S^*(E_i)) = \sum g((S + S^*)(E_i), D_X E_i) \]

\[
= \sum_{i,j} g((S + S^*)(E_i), E_j)g(E_j, D_X E_i) = \sum_{i<j} g((S + S^*)(E_i), E_j)g(E_j, D_X E_i) + \sum_{i>j} g((S + S^*)(E_i), E_j)g(E_j, D_X E_i) \]

\[
= \sum_{i<j} g((S + S^*)(E_i), E_j)g(E_j, D_X E_i) + \sum_{i<j} g((S + S^*)(E_j), E_i)g(E_i, D_X E_i) - \sum_{i<j} g((S + S^*)(E_i), E_j)g(E_j, D_X E_i) \]

\[
= \sum_{i<j} g((S + S^*)(E_i), E_j)g(E_j, D_X E_i) \quad \text{by (3)}
\]

\[
= 0. \quad \Box
\]

**Theorem 6 (Bochner 1964).** If $(M, g)$ is compact, oriented, and has $\text{Ric} \geq 0$, then every harmonic 1-form is parallel.

**Proof.** Suppose $\omega$ is a harmonic 1-form, $X$ the dual vector field, and $f = \frac{1}{2}g(\omega, \omega) = \frac{1}{2}|X|^2$. Then

\[
\Delta f = |DX|^2 + \text{Ric}(X, X) \quad (\because \text{div} X = 0).
\]

\[
\therefore 0 = \int_M \Delta f \cdot dv_g = \int_M (|DX|^2 + \text{Ric}(X, X)) \cdot dv_g \geq \int_M |DX|^2 \cdot dv_g \geq 0.
\]

We can therefore conclude that $|DX| = 0$. \quad \Box
Corollary 7. If \((M, g)\) is compact, oriented, has \(\text{Ric} \geq 0\), and has positive Ricci curvature at one point, then all harmonic 1-form vanish everywhere.

Proof. From the proof of Theorem 6, we obtain \(\text{Ric}(X, X) \equiv 0\), under the assumption that \((M, g)\) is compact, oriented, and has \(\text{Ric} \geq 0\). If the Ricci tensor is positive on \(T_pM\), we must have \(X(p) = 0\).
But then \(X \equiv 0\), since \(X\) is parallel. \(\square\)

Corollary 8. If \((M, g)\) is compact, oriented, and has \(\text{Ric} \geq 0\), then

\[
\dim \mathbb{H}^1(M) \leq n.
\]

Proof. All the harmonic 1-forms are parallel, and hence is determined by its value at one point. Therefore the linear map \(\mathbb{H}^1(M) \to T_pM\) defined by \(\omega \mapsto X(p)\) is injective. \(\square\)

Corollary 9. If \((M, g)\) is compact, oriented, and has \(\text{Ric} \geq 0\), then

\[
b_1(M) \leq n = \dim M,
\]
with equality holding iff \((M, g)\) is a flat torus.

Proof. We know from the Hodge theorem that

\[
b_1(M) = \dim \mathbb{H}^1(M).
\]

Thus (4) follows from Corollary 8.

- If equality holds, we obviously have \(n\) linearly independent parallel fields \(E_i\), \(i = 1, \ldots, n\).
  This clearly implies that \((M, g)\) is flat.
  Thus the universal covering is \((\mathbb{R}^n, \text{can})\) with \(\Gamma = \pi_1(M)\) acting by isometries.
  To complete the proof, it suffices to
  Claim: (i) \(\Gamma\) consists entirely of translations; i.e. \(\Gamma\) is finitely generated, abelian, with no fixed point, and hence must be \(\mathbb{Z}^q\) for some \(q\).
  (ii) \(q = n\).
  (i) Pull the vector fields \(E_i\) back to \(\tilde{E}_i\), \(i = 1, \ldots, n\), on \(\mathbb{R}^n\).
  These vector fields \(\tilde{E}_i\) are again parallel and are therefore constant vector fields.
  - In addition, they are invariant under the action of \(\Gamma\); i.e., for any \(\gamma \in \Gamma\), we have
    \[
    D\gamma(\tilde{E}_i(p)) = \tilde{E}_i(\gamma(p)), \quad i = 1, \ldots, n.
    \]
    A basis of constant vector fields can be invariant only under translations.
  (ii) If \(q < n\), then \(\mathbb{Z}^q\) generates a subspace \(V\) of \(\mathbb{R}^n\) with dimension \(< n\).
  Let \(W = v^\perp\), the orthogonal complement of \(V\) in \(\mathbb{R}^n\).
  Then
  \[
  M = \mathbb{R}^n/\mathbb{Z}^q = (V \oplus W)/\mathbb{Z}^q = (v/\mathbb{Z}^q) \oplus W,
  \]
  which is not compact and is a contradiction. \(\square\)