Chern Classes

(a) Connection and Curvature in a Complex Vector Bundle.

- Suppose $\pi : E \to M$ is an n-dimensional complex vector bundle.
  The set of all sections $\Gamma(E)$ is not only a module over the ring of all real-valued functions $C^\infty(M)$
  but also a module over the ring of all complex-valued functions $C^\infty(M)$, which we denote by $C^\infty(M; \mathbb{C}) = C^\infty(M) \otimes \mathbb{C}$.
- Also, for differential forms we set

$$\mathcal{A}^k(M; \mathbb{C}) = \mathcal{A}^k(M) \otimes \mathbb{C}$$

  and call its elements **complex k-forms**.

– By definition, an arbitrary complex k-forms can be uniquely written in the form

$$\omega + i\eta, \quad \text{where } \omega,\, \eta \in \mathcal{A}^k(M), \text{ and } i \text{ is the imaginary unit.}$$

- Exterior differentiation

$$\mathcal{A}^k(M; \mathbb{C}) \to \mathcal{A}^{k+1}(M; \mathbb{C})$$

  is defined by simply extending ordinary d linealy over $\mathbb{C}$.
- The cochain comples $\{\mathcal{A}^k(M; \mathbb{C}); d\}$ is called the comlex de Rham complex and the complex de Rham complex and its cohomology is denoted by $H^*_{dR}(M; \mathbb{C})$.

$$H^*_{dR}(M; \mathbb{C}) = H^*_{dR}(M) \otimes \mathbb{C} \cong H^*(M; \mathbb{C}).$$

Definition. *Given a complex vector bundle $E \to M$, a* **connection** *is a connection*

$$D : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

for the underlying real vector bundle E that furthermore satisfies the condition

$$D_X(is) = iD_X s,$$

which is equivalent to the condition

$$D_X(fs) = (Xf)s + fD_X s, \quad \forall f \in C^\infty(M; \mathbb{C}), \quad \forall s \in \Gamma(E).$$

- If we use the description in terms of differential forms with values in a vector bundle, we can say that a connection $D : \Gamma(E) \to \mathcal{A}^1(M; E)$ is a complex linear map such that

$$D(fs) = df \otimes s + fDs, \quad \forall f \in C^\infty(M; \mathbb{C}), \quad \forall s \in \Gamma(E).$$

- The curvature of a connection in a complex vector bundle is defined accordingly, using the same formula aas in the case of a real vector bundle.

- Let us now consider the connection form and curvature form.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

– If, in an open subset U, we are given a frame field $s_1, \cdots, s_n \in \Gamma(E\Big|_U)$, then writing

$$D_X s_j = \sum_{i=1}^{n} \omega_j^i(X) s_i, \quad \forall X \in \Gamma(TU),$$

we obtain complex 1-forms $\omega_j^i \in \mathcal{A}^1(U;\mathbb{C})$ over U.
Put them together, $\omega = (\omega_j^i)$, and we obtain a 1-fom on U with values in $\mathrm{M}(n:\mathbb{C})$; we call it **the connection form for** D.

– Similarly, we can obtain the curvature fom $\Omega = (\Omega_j^i)$, as a 2-form with values in $\mathrm{M}(n:\mathbb{C})$.

– The structure equation and the Bianchi identity hold in the same form.
The tansformation formula for the connection and curvature forms remain the same except that the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n,\mathbb{C})$ now have values in $\mathrm{GL}(n,\mathbb{C})$.

(b) Definition of Chern Classes.

- A connection D is a complex bundle $\pi : E \to M$ leads to curvature R; locally, we have the connection form $\omega = (\omega_j^i)$ and the curvature form $\Omega = (\Omega_j^i)$, which are related by the transition functions $g_{\alpha\beta}$, namely

$$\Omega_\beta = g_{\alpha\beta}^{-1} \Omega_\alpha g_{\alpha\beta},$$

in any non-empty intersection $U_\alpha \cap U_\beta$.

- In order to construct a differential forms globally on M, we make the same definition as in the case of a real vector bundle. Namely, a polynomial function

$$f : \mathrm{M}(n;\mathbb{C}) \to \mathbb{C}$$

such that

$$f(X) = f(A^{-1}XA), \quad \forall A \in \mathrm{GL}(n;\mathbb{C})$$

is called an **invariant polynomial function** on $\mathrm{GL}(n;\mathbb{C})$.

– The set of all invariant polynomials is denoted by $I_n(\mathbb{C})$.

– There is an isomophism

$$I_n(\mathbb{C}) \cong S_n(\mathbb{C}),$$

where $S_n(\mathbb{C})$ is the commutative algebra of all symmetric polynomials with complex coefficients in n variables.

- By a parallel process we can prove the following results.

(i) For any invariant polynomial $f \in I_n(\mathbb{C})$ of degree k, we have
  (1) $f(\Omega) \in \mathcal{A}^{2k}(M;\mathbb{C})$;
  (2) $f(\Omega)$ is a closed form;
  (3) the corresponding de Rham cohomology class $[f(\Omega)] \in H^{2k}(M;\mathbb{C})$ is determined independently of the choice of the connection.

Definition. *The de Rham cohomology class* $[f(\Omega)]$ *is called the* **characteristic class** *of* E *corresponding to* f *and denote it by* $f(E)$.

- The characteristic class is **natural** with respect to bundle maps; namely, for any C^∞ map, $g : N \to M$, g^*E is the induced bundle and

$$f(g^*E) = g^*(f(E)) \in H^{2k}(N, \mathbb{C}).$$

- The situation of a complex vector bundle is entirely different from the real case in that the characteristic class corresponding to an invariant polynomial of **odd degree** is **not** trivial.

Definition. *For an* n*-dimensional complex vector bundle* $\pi : E \to M$*, the characteristic class that corresponds to the invariant polynomial*

$$\frac{1}{(2\pi i)^k}\sigma_k \in I_n(\mathbb{C})$$

is written $c_k(E) \in H^{4k}_{dR}(M)$*, and is called the* **Chern class** *of degree* k.

- *In terms of the curvature fom* Ω*, we may write*

$$c(E) \overset{\text{def}}{=} \left[\det\left(1 - \frac{1}{2\pi i}\Omega\right)\right] = 1 + c_1(E) + c_2(E) + \cdots + c_n(E),$$

  and call it the **total Chern class**.
- *The closed form representing the Chern class that corresponds to any particular connection is called the* **Chern form**.

Proposition 1. *Each Chern class* c_k *is a real cohomology class;*
i.e. $c_k(E) \in H^{4k}_{dR}(M) = H^{2k}(M; \mathbb{R})$ *and* $c(E) \in H^*_{dR}(M)$.

Proof. We introduce a Hermitian metric on E; recall that a Hermitian metric is positive-definite in each fiber E_p, which is conjugate linear in the first component, namely

$$\langle av, bv' \rangle = \overline{a}b\langle v, v' \rangle, \quad \forall a, b \in \mathbb{C}, \quad v, v' \in E_p.$$

Then we construct a connection that is compatible with the metric, that is,

$$X\langle s, s' \rangle = \langle D_X s, s' \rangle + \langle s, D_X s' \rangle, \quad \forall X \in \Gamma(E), \quad s, s' \in \Gamma(E);$$

this can be done by an argument similar to the proof in the real case.

– It is also easy to show that the corresponding connection form $\omega = (\omega^i_j)$ and the curvature form $\Omega = (\Omega^i_j)$ are both skew-Hermitian, namely,

$$\omega^i_j + \overline{\omega}^j_i = 0, \quad \Omega^i_j + \overline{\Omega}^j_i = 0.$$

– Now if X is a skew-Hermitian matrix, then $I - \frac{1}{2\pi i}X$ is a Hermitian matrix, so that its determinant is a real number. $\square$

(c) Whitney Formula.

Definition. *Suppose two vector bundles $\pi_i : E_i \to M_i$, $i = 1, 2$, over the same base space are given. Then the set*

$$E_1 \oplus E_2 = \{(u_1, u_2) \in E_1 \times E_2;\ \pi_1(u_1) = \pi_2(u_2)\}$$

with the projection

$$\pi : E_1 \oplus E_2 \ni (u_1, u_2) \mapsto \pi_1(u_1) \in M$$

is the **Whitney sum** *of E_1 and E_2.*

- We have dim $(E_1 \oplus E_2)$ =dim E_1+dim E_2.

Example. Let E be a vector bundle and F an arbitrary subbundle. Then there is an isomorphism $E \cong F \oplus E/F$.

- The charcteristic classes of the Whitney sum of two vector bundles is given by the following Whitnel formula.

Theorem 2. *(i) If E and F are complex vector bundles, then*

$$c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E) c_{k-i}(F),$$

i.e., $c(E \oplus F) = c(E)c(F)$.
(ii) If E and F are real vector bundles, then

$$p_k(E \oplus F) = \sum_{i=0}^{k} p_i(E) p_{k-i}(F),$$

i.e., $p(E \oplus F) = p(E)p(F)$.

Proof. (i) Clearly, $\Gamma(E \oplus F) = \Gamma(E) \times \Gamma(F)$.
It follows that if D and D' are the connections of E and F, then there is a natural direct sum connection $D \oplus D'$ on $E \oplus F$.
if Ω and Ω' are the curvature forms of D and D', then the curvature form $\widetilde{\Omega}$ of $D \oplus D'$ is the direct sum matrix of Ω and Ω':

$$\widetilde{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega' \end{pmatrix}.$$

Hence

$$\begin{aligned} c(E \oplus F) &= \det\left[1 - \frac{1}{2\pi i}\widetilde{\Omega}\right] \\ &= \det\left[1 - \frac{1}{2\pi i}\Omega\right] \det\left[1 - \frac{1}{2\pi i}\Omega'\right]. \end{aligned}$$

(ii) The proof of (ii) is similar. $\square$

(d) Relations between Pontryagin and Chern Classes.

- If E is an n-dimensional real vector budle, its Pontrjagin class $p(E) \subset H^*(M, \mathbb{R})$ is defined.
- On the other hand, since the complexification $E \oplus \mathbb{C}$ of E is an n-dimensional complex vector bundle, its chern class $c(E \oplus \mathbb{C}) \in H^*(M; \mathbb{R})$ is defined.
- There is a close relationship between these characteristic classes.

Proposition 3. *Let E be a real vector bundle and $E \otimes C$ its complexification. Then*

$$p_k(E) = (-1)^k c_{2k}(E \otimes C) \in H^{2k}(M; \mathbb{Z}).$$

Proof. Our proof depends on using differential forms and is limited to the eal case.
- A connection D in E naturally induces the connection $D \otimes \mathbb{C}$.
- The connection and curvature forms ω in Ω extend to the correspnding forms for $D \otimes \mathbb{C}$. Therefoe we have

$$\begin{aligned} p_k(E) =& \left[\left(\frac{1}{2\pi} \right)^{2k} \sigma_{2k}(\Omega) \right] = (-1)^k \left[\left(\frac{1}{2\pi i} \right)^{2k} \sigma_{2k}(\Omega) \right] \\ =& (-1)^k c_{2k}(E \otimes \mathbb{C}). \qquad \square \end{aligned}$$

- Next let E be an n-dimensional complex vector bundle.
  We may think of it as a $2n$-dimensional real vector bundle.
  How are the Chern classes of E and the Pontrjagin classes of E related to each other?

Definition. *Let E be an n-dimensional complex vector bundle. On each fiber E_p, $p \in M$, we define multiplication by a complex number $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$, by setting*

$$(a + bi)v = av - biv, \quad \forall v \in E_p.$$

We take $\overline{E}_p$ as a new complex vector space and define

$$\overline{E} = \bigcup \overline{E}_p$$

as the **conjugate bundle**.

Lemma 4. *The conjugate bundle $\overline{E}$ of a complex vector bundle E is isomorphic to the dual bundle E^* of E.*

Proof. Introduce a Hermitian metric in E and consider on each fiber the map

$$\begin{aligned} v \in \overline{E}_p \mapsto& \ell(v) \in E_p^*; \\ \ell(v) : u \in E_p \mapsto& \ell(v)u = \langle v, u \rangle \in \mathbb{C}. \end{aligned}$$

It is easy to verify that we obtain an isomorphism $\overline{E} \cong E^*$. $\square$

Proposition 5. *The Chern classes of the conjugate bundle E of a complex vector bundle are given by*

$$c_k(\overline{E}) = (-1)^k c_k(E). \tag{1}$$

We have also for the dual bundle

$$c_k(E^*) = (-1)^k c_k(E). \tag{2}$$

Proof. (1) A connection D for E remains a connection for $\overline{E}$.
- If D has the curvature form Ω, then D for $\overline{E}$ has $\overline{\Omega}$ as curvature form.
- On the other hand, we may assume that Ω is skew-Hermitian from the Proposition 1. Hence $\overline{\Omega} = -{}^t\Omega$.
- By putting this into the definition of the Chern class, we obtain the formula we want.

(2) Combining (1) with Lemma 4, we obtain the second formula. □

Proposition 6. *Let E be an n-dimensional complex vector bundle. Then, writing p_i for $p_i(E)$ and c_i for $c_i(E)$, we have*

$$\begin{aligned}1 - p_1 + p_2 - \cdots + (-1)^n p_n \\ = (1 + c_1 + c_2 + \cdots + c_n)(1 - c_1 + c_2 - \cdots + (-1)^n c_n)\end{aligned}$$

For example, we have

$$p_1 = c_1^2 - 2c_2; \quad p_2 = c_2^2 - 2c_1c_3 + 2c_4.$$

Proof. **Step 1.** We write $E_{\mathbb{R}}$ when E is regarded as a real vector bundle.
Then $E_{\mathbb{R}} \otimes \mathbb{C}$ is a $2n$-dimensional complex vector bundle, and there is a netural isomorphism

$$E_{\mathbb{R}} \otimes \mathbb{C} \cong E \otimes \overline{E}.$$

To see this, we consider the correspondence for each fiber

$$(E_{\mathbb{R}} \otimes \mathbb{C})_p \ni u + v \otimes i \mapsto \left(\frac{u+iv}{2}, \frac{u-iv}{2}\right) \in E_p \oplus \overline{E}_p, \quad (u, v, iv \in E_p = \overline{E}_p),$$

which nduces an isomorphism over $\mathbb{C}$ since

$$i(u + v \otimes i) = -v + u \otimes i \mapsto \left(\frac{-v+iu}{2}, \frac{-v-iu}{2}\right) = i\left(\frac{u+iv}{2}, \frac{u-iv}{2}\right).$$

Step 2. By Proposition 3, we have $p_k = (-1)^k c_{2k}(E_{\mathbb{R}} \otimes C)$.
On the other hand, by applying the Whitney formula to the isomorphism above, we obtain

$$c(E_{\mathbb{R}} \otimes \mathbb{C}) = c(E \oplus \overline{E}) = c(E)c(\overline{E}).$$

Now we can complete the proof by using Proposition 5. □