Chern Classes

(a) Connection and Curvature in a Complex Vector Bundle.

• Suppose \( \pi : E \to M \) is an \( n \)-dimensional complex vector bundle. The set of all sections \( \Gamma(E) \) is not only a module over the ring of all real-valued functions \( C^\infty(M) \) but also a module over the ring of all complex-valued functions \( C^\infty(M) \), which we denote by \( C^\infty(M; \mathbb{C}) = C^\infty(M) \otimes \mathbb{C} \).

• Also, for differential forms we set \( \mathcal{A}^k(M; \mathbb{C}) = \mathcal{A}^k(M) \otimes \mathbb{C} \) and call its elements complex \( k \)-forms.

  – By definition, an arbitrary complex \( k \)-forms can be uniquely written in the form \( \omega + i\eta \), where \( \omega, \eta \in \mathcal{A}^k(M) \), and \( i \) is the imaginary unit.

• Exterior differentiation
  \[ \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M; \mathbb{C}) \]

  is defined by simply extending ordinary \( d \) linealy over \( \mathbb{C} \).

• The cochain complexes \( \{ \mathcal{A}^k(M; \mathbb{C}); d \} \) is called the complex de Rham complex and its cohomology is denoted by \( H^*_dR(M; \mathbb{C}) \).

\[ H^*_dR(M; \mathbb{C}) = H^*_dR(M) \otimes \mathbb{C} \cong H^*(M; \mathbb{C}) \].

**Definition.** Given a complex vector bundle \( E \to M \), a **connection** is a connection

\[ D : \Gamma(TM) \times \Gamma(E) \to \Gamma(E) \]

for the underlying real vector bundle \( E \) that furthermore satisfies the condition

\[ D_X(is) = iD_Xs, \]

which is equivalent to the condition

\[ D_X(fs) = (Xf)s + fD_Xs, \quad \forall f \in C^\infty(M; \mathbb{C}), \quad \forall s \in \Gamma(E). \]

• If we use the description in terms of differential forms with values in a vector bundle, we can say that a connection \( D : \Gamma(E) \to \mathcal{A}^1(M; E) \) is a complex linear map such that

\[ D(fs) = df \otimes s + fDs, \quad \forall f \in C^\infty(M; \mathbb{C}), \quad \forall s \in \Gamma(E). \]

• The curvature of a connection in a complex vector bundle is defined accordingly, using the same formula as in the case of a real vector bundle.

• Let us now consider the connection form and curvature form.
If, in an open subset $U$, we are given a frame field $s_1, \ldots, s_n \in \Gamma(E|_U)$, then writing
\[ D_X s_j = \sum_{i=1}^{n} \omega^i_j(X)s_i, \quad \forall X \in \Gamma(TU), \]
we obtain complex 1-forms $\omega^j_j \in A^1(U; \mathbb{C})$ over $U$.

Put them together, $\omega = (\omega^j_j)$, and we obtain a 1-form on $U$ with values in $M(n : \mathbb{C})$; we call it the connection form for $D$.

Similarly, we can obtain the curvature form $\Omega = (\Omega^j_j)$, as a 2-form with values in $M(n : \mathbb{C})$.

The structure equation and the Bianchi identity hold in the same form.

The transformation formula for the connection and curvature forms remain the same except that the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}(n, \mathbb{C})$ now have values in $\text{GL}(n, \mathbb{C})$.

(b) Definition of Chern Classes.

- A connection $D$ is a complex bundle $\pi : E \to M$ leads to curvature $R$; locally, we have the connection form $\omega = (\omega^i_j)$ and the curvature form $\Omega = (\Omega^i_j)$, which are related by the transition functions $g_{\alpha\beta}$, namely
\[ \Omega^j_j = g^{-1}_{\alpha\beta} \Omega^j_j g_{\alpha\beta}, \]
in any non-empty intersection $U_\alpha \cap U_\beta$.

- In order to construct a differential forms globally on $M$, we make the same definition as in the case of a real vector bundle. Namely, a polynomial function
\[ f : M(n; \mathbb{C}) \to \mathbb{C} \]
such that
\[ f(X) = f(A^{-1}XA), \quad \forall A \in \text{GL}(n; \mathbb{C}) \]
is called an invariant polynomial function on $\text{GL}(n; \mathbb{C})$.

- The set of all invariant polynomials is denoted by $I_n(\mathbb{C})$.

- There is an isomorphism
\[ I_n(\mathbb{C}) \cong S_n(\mathbb{C}), \]
where $S_n(\mathbb{C})$ is the commutative algebra of all symmetric polynomials with complex coefficients in $n$ variables.

- By a parallel process we can prove the following results.

(i) For any invariant polynomial $f \in I_n(\mathbb{C})$ of degree $k$, we have
   1. $f(\Omega) \in A^{2k}(M; \mathbb{C})$;
   2. $f(\Omega)$ is a closed form;
   3. the corresponding de Rham cohomology class $[f(\Omega)] \in H^{2k}(M; \mathbb{C})$ is determined independently of the choice of the connection.

Definition. The de Rham cohomology class \( f(\Omega) \) is called the characteristic class of \( E \) corresponding to \( f \) and denote it by \( f(E) \).

- The characteristic class is natural with respect to bundle maps; namely, for any \( C^\infty \) map, \( g : N \to M \), \( g^*E \) is the induced bundle and
  \[
  f(g^*E) = g^*(f(E)) \in H^{2k}(N, \mathbb{C}).
  \]

- The situation of a complex vector bundle is entirely different from the real case in that the characteristic class corresponding to an invariant polynomial of odd degree is not trivial.

Definition. For an \( n \)-dimensional complex vector bundle \( \pi : E \to M \), the characteristic class that corresponds to the invariant polynomial
  \[
  \frac{1}{(2\pi i)^k} \sigma_k \in I_n(\mathbb{C})
  \]
is written \( c_k(E) \in H^{4k}_{dR}(M) \), and is called the Chern class of degree \( k \).

- In terms of the curvature form \( \Omega \), we may write
  \[
  c(E) \overset{\text{def}}{=} \det \left( 1 - \frac{1}{2\pi i} \Omega \right) = 1 + c_1(E) + c_2(E) + \cdots + c_n(E),
  \]
  and call it the total Chern class.

- The closed form representing the Chern class that corresponds to any particular connection is called the Chern form.

Proposition 1. Each Chern class \( c_k \) is a real cohomology class; i.e. \( c_k(E) \in H^{4k}_{dR}(M) = H^{2k}(M; \mathbb{R}) \) and \( c(E) \in H^{*}_{dR}(M) \).

Proof. We introduce a Hermitian metric on \( E \); recall that a Hermitian metric is positive-definite in each fiber \( E_p \), which is conjugate linear in the first component, namely
  \[
  \langle av, bv' \rangle = \overline{\langle v, v' \rangle}, \quad \forall a, b \in \mathbb{C}, \ v, v' \in E_p.
  \]
Then we construct a connection that is compatible with the metric, that is,
  \[
  X (s, s') = \langle D_X s, s' \rangle + \langle s, D_X s' \rangle, \quad \forall X \in \Gamma(E), \ s, s' \in \Gamma(E);
  \]
this can be done by an argument similar to the proof in the real case.

- It is also easy to show that the corresponding connection form \( \omega = (\omega^i_j) \) and the curvature form \( \Omega = (\Omega^i_j) \) are both skew-Hermitian, namely,
  \[
  \omega^i_j + \overline{\omega^i_j} = 0, \quad \Omega^i_j + \overline{\Omega^i_j} = 0.
  \]

- Now if \( X \) is a skew-Hermitian matrix, then \( I - \frac{1}{2\pi i} X \) is a Hermitian matrix, so that its determinant is a real number. \( \square \)
(c) Whitney Formula.

**Definition.** Suppose two vector bundles \( \pi_i : E_i \to M_i, \ i = 1, 2, \) over the same base space are given. Then the set

\[
E_1 \oplus E_2 = \{(u_1, u_2) \in E_1 \times E_2; \ \pi_1(u_1) = \pi_2(u_2)\}
\]

with the projection

\[
\pi : E_1 \oplus E_2 \ni (u_1, u_2) \mapsto \pi_1(u_1) \in M
\]

is the **Whitney sum** of \( E_1 \) and \( E_2 \).

- We have \( \dim (E_1 \oplus E_2) = \dim E_1 + \dim E_2 \).

**Example.** Let \( E \) be a vector bundle and \( F \) an arbitrary subbundle. Then there is an isomorphism \( E \cong F \oplus E/F \).

- The characteristic classes of the Whitney sum of two vector bundles is given by the following Whitney formula.

**Theorem 2.** (i) If \( E \) and \( F \) are complex vector bundles, then

\[
c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E)c_{k-i}(F),
\]

i.e., \( c(E \oplus F) = c(E)c(F) \).

(ii) If \( E \) and \( F \) are real vector bundles, then

\[
p_k(E \oplus F) = \sum_{i=0}^{k} p_i(E)p_{k-i}(F),
\]

i.e., \( p(E \oplus F) = p(E)p(F) \).

**Proof.** (i) Clearly, \( \Gamma(E \oplus F) = \Gamma(E) \times \Gamma(F) \).

It follows that if \( D \) and \( D' \) are the connections of \( E \) and \( F \), then there is a natural direct sum connection \( D \oplus D' \) on \( E \oplus F \).

If \( \Omega \) and \( \Omega' \) are the curvature forms of \( D \) and \( D' \), then the curvature form \( \tilde{\Omega} \) of \( D \oplus D' \) is the direct sum matrix of \( \Omega \) and \( \Omega' \):

\[
\tilde{\Omega} = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega' \end{pmatrix}.
\]

Hence

\[
c(E \oplus F) = \det \left[ 1 - \frac{1}{2\pi i} \tilde{\Omega} \right] = \det \left[ 1 - \frac{1}{2\pi i} \Omega \right] \det \left[ 1 - \frac{1}{2\pi i} \Omega' \right].
\]

(ii) The proof of (ii) is similar. □
(d) Relations between Pontryagin and Chern Classes.

- If $E$ is an $n$-dimensional real vector bundle, its Pontrjagin class $p(E) \subset H^*(M, \mathbb{R})$ is defined.
  - On the other hand, since the complexification $E \oplus \mathbb{C}$ of $E$ is an $n$-dimensional complex vector bundle, its Chern class $c(E \oplus \mathbb{C}) \in H^*(M; \mathbb{R})$ is defined.
  - There is a close relationship between these characteristic classes.

**Proposition 3.** Let $E$ be a real vector bundle and $E \otimes \mathbb{C}$ its complexification. Then

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{2k}(M; \mathbb{Z}).$$

**Proof.** Our proof depends on using differential forms and is limited to the real case.
  - A connection $D$ in $E$ naturally induces the connection $D \otimes \mathbb{C}$.
  - The connection and curvature forms $\omega$ in $\Omega$ extend to the corresponding forms for $D \otimes \mathbb{C}$. Therefore we have

$$p_k(E) = \left(\frac{1}{2\pi}\right)^{2k} \sigma_{2k}(\Omega) = (-1)^k \left(\frac{1}{2\pi i}\right)^{2k} \sigma_{2k}(\Omega)$$

$$= (-1)^k c_{2k}(E \otimes \mathbb{C}). \qed$$

- Next let $E$ be an $n$-dimensional complex vector bundle. We may think of it as a $2n$-dimensional real vector bundle.

**How are the Chern classes of $E$ and the Pontrjagin classes of $E$ related to each other?**

**Definition.** Let $E$ be an $n$-dimensional complex vector bundle. On each fiber $E_p$, $p \in M$, we define multiplication by a complex number $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$, by setting

$$(a + bi)v = av - biv, \quad \forall v \in E_p.$$ 

We take $E_p$ as a new complex vector space and define

$$E = \bigcup E_p$$

as the conjugate bundle.

**Lemma 4.** The conjugate bundle $E$ of a complex vector bundle $E$ is isomorphic to the dual bundle $E^*$ of $E$.

**Proof.** Introduce a Hermitian metric in $E$ and consider on each fiber the map

$$v \in E_p \mapsto \ell(v) \in E_p^*;$$

$$\ell(v) : u \in E_p \mapsto \ell(v)u = \langle v, u \rangle \in \mathbb{C}.$$ 

It is easy to verify that we obtain an isomorphism $E \cong E^*$. \qed
Proposition 5. The Chern classes of the conjugate bundle \( E \) of a complex vector bundle are given by

\[
(1) \quad c_k(\overline{E}) = (-1)^k c_k(E).
\]

We have also for the dual bundle

\[
(2) \quad c_k(E^*) = (-1)^k c_k(E).
\]

Proof. (1) A connection \( D \) for \( E \) remains a connection for \( \overline{E} \).

- If \( D \) has the curvature form \( \Omega \), then \( D \) for \( \overline{E} \) has \( \overline{\Omega} \) as curvature form.
- On the other hand, we may assume that \( \Omega \) is skew-Hermitian from the Proposition 1. Hence \( \overline{\Omega} = -i \Omega \).
- By putting this into the definition of the Chern class, we obtain the formula we want.

(2) Combining (1) with Lemma 4, we obtain the second formula. □

Proposition 6. Let \( E \) be an \( n \)-dimensional complex vector bundle. Then, writing \( p_i \) for \( p_i(E) \) and \( c_i \) for \( c_i(E) \), we have

\[
1 - p_1 + p_2 - \cdots + (-1)^n p_n = (1 + c_1 + c_2 + \cdots + c_n)(1 - c_1 + c_2 - \cdots + (-1)^n c_n)
\]

For example, we have

\[
p_1 = c_1^2 - 2c_2; \quad p_2 = c_2^2 - 2c_1c_3 + 2c_4.
\]

Proof. Step 1. We write \( E_\mathbb{R} \) when \( E \) is regarded as a real vector bundle.

Then \( E_\mathbb{R} \otimes \mathbb{C} \) is a \( 2n \)-dimensional complex vector bundle, and there is a natural isomorphism

\[
E_\mathbb{R} \otimes \mathbb{C} \cong E \otimes \overline{E}.
\]

To see this, we consider the correspondence for each fiber

\[
(E_\mathbb{R} \otimes \mathbb{C})_p \ni u + v \otimes i \mapsto \left( \frac{u + iv}{2}, \frac{u - iv}{2} \right) \in E_p \oplus \overline{E}_p, \quad (u, v, iv, u + iv, u - iv) \in \mathbb{C},
\]

which induces an isomorphism over \( \mathbb{C} \) since

\[
i(u + v \otimes i) = -v + u \otimes i \mapsto \left( \frac{-v + iv}{2}, \frac{-v - iv}{2} \right) = i \left( \frac{u + iv}{2}, \frac{u - iv}{2} \right).
\]

Step 2. By Proposition 3, we have \( p_k = (-1)^k c_{2k}(E_\mathbb{R} \otimes \mathbb{C}) \).

On the other hand, by applying the Whitney formula to the isomorphism above, we obtain

\[
c(E_\mathbb{R} \otimes \mathbb{C}) = c(E \oplus \overline{E}) = c(E)c(\overline{E}).
\]

Now we can complete the proof by using Proposition 5. □