Dirac Operators

**Definition.** The principal symbol of a differential operator $P : \Gamma(E) \to \Gamma(E)$ is a map which associates each point $x \in X$ and each cotangent vector $\xi \in T^*_x(X)$ a linear map $\sigma_\xi(P) : E_x \to E_x$ defined as follows. If in local coordinates we have

$$P = \sum_{|\alpha| \leq m} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

and $\xi = \sum_k \xi_k dx_k$, where $m$ is the order of $P$,

then

$$\sigma_\xi(P)(x;\xi) = (-i)^m \sum_{|\alpha| = m} A_\alpha(x) \xi_\alpha,$$

is called the principal symbol of $P$.

**Definition.** Let $P$ be a differential operator of order $m$ on a manifold $M$; $P$ could operate on sections of a vector bundle. In local coordinatea, $P$ has the form

$$Pu(x) = \sum_{|\alpha| \leq m} p_\alpha(x) D^\alpha u(x),$$

where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = (1/i) \partial/\partial x_j$. The coefficients $p_\alpha(x)$ could be matrix valued. The homogeneous polynomial in $\xi \in \mathbb{R}^n$ ($n = \dim M$),

$$p(x;\xi) = \sum_{|\alpha| = m} p_\alpha(x) \xi_\alpha$$

is called the principal symbol of $P$.

**Definition.** The operator $P$ is elliptic if $\sigma_\xi(P;\xi)$ is an isomorphism for all $\xi \neq 0$, i.e. if $\sigma_\xi(P)$ is invertible for all $\xi \neq 0$.

**Definition.** The operator $P$ is elliptic if, for some $r < \infty$ and $C \in \mathbb{R}$,

$$|p(x,\xi)|^{-1} \leq C|\xi|^{-m}, \quad \text{for } |\xi| \geq r.$$

**Definition.** Let $M$ be a Riemannian manifold, $E_j \to M$ a vector bundle with Hermitian metrics. A first-order, elliptic differential operator

$$P : C^\infty(M, E_0) \to C^\infty(M, E_1)$$

is said to be of Dirac type if $P^*P$ has scalar principal symbol. This implies

$$\sigma_{P^*P}(x,\xi) = q(x,\xi) I : E_{0x} \to E_{0x},$$

where $q(x,\xi)$ is a quadratic form on $M$.

- If $E_0 = E_1$ and $P = P^*$, we say $P$ is a symmetric Dirac-type operator.
- Given a general operator $D$ of Dirac type, if we set $E = E_0 \oplus E_1$ and define $\tilde{P}$ on $C^\infty(M, E)$ as

$$(3) \quad \tilde{P} = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix},$$

then $\tilde{P}$ is a symmetric Dirac-type operator.
• We want to give an intrinsic characterization, which will show that $p_m(x, \xi)$ is well-defined on $T^*M$.

- For a smooth function $\psi$, using product rule and chain rule of differentiation, we have

\[
P(u(x)e^{i\lambda \psi}) = [p_m(x, d\psi)u(x)\lambda^m + r(x, \lambda)]e^{i\lambda \psi}
\]

where $r(x, \lambda)$ is a polynomial of degree $\leq m - 1$.

- In (4), $p_m(x, d\psi)$ is evaluated by substituting $\xi = (\partial \psi/\partial x_1, \ldots, \partial \psi/\partial x_n)$ into (2). Thus the formula

\[
p_m(x, d\psi)u(x) = \lim_{\lambda \to \infty} \lambda^{m-1}e^{-i\lambda \psi} P[u(x)e^{i\lambda \psi}]
\]

provides an intrinsic characterization of the symbol of $P$ as a function on $T^*M$.

• If $M$ has a Riemannian metric, and the vector bundle $E_j$ have metrics, then the formal adjoint $P^*$ of a differential operator of order $m$ is a differential operator of order $m$:

\[
P^*: C^\infty(M, E_1) \to C^\infty(M, E_0),
\]

defined by the condition that

\[
(Pu, v) = (u, P^*v)
\]

if $u$ and $v$ are smooth, compactly supported sections of the bundles $E_0$ and $E_1$.

• If $U$ and $V$ are supported on a coordinate patch $O$ on $M$, over which $E_j$ are trivialized, so $u$ and $v$ have components $u^\sigma$, $v^\sigma$, and the Riemannian metrics on $E_0$ and $E_1$ are denoted $h_{\sigma\delta}$, $\tilde{h}_{\sigma\delta}$, respectively, while the Riemannian metric on $M$ is $g_{jk}$, then we have

\[
(Pu, v) = \int_O \tilde{h}_{\sigma\delta}(Pu)^\sigma \sqrt{g(x)} dx.
\]

Integrating by parts produces an expression for $P^*$ of the form

\[
P^*v(x) = \sum_{|\alpha|\leq m} p^T_\alpha(x)D^\alpha v(x).
\]

In particular, one sees that the principal symbol of $P^*$ is given by

\[
\sigma_{P^*}(x, \xi) = \sigma_P(x, \xi)^*.
\]
Let $\vartheta(x, \xi)$ denote the principal symbol of a symmetric Dirac-type operator. With $x \in M$ fixed, set $\vartheta(\xi) = \vartheta(x, \xi)$. Thus $\vartheta$ is a linear map from $T_x^*M = \{\xi\}$ into $\text{End}(E_x)$, satisfying

\begin{equation}
\vartheta(\xi) = \vartheta(\xi)^*
\end{equation}

and

\begin{equation}
\vartheta(\xi)^2 = \langle \xi, \xi \rangle I.
\end{equation}

Here $\langle , \rangle$ is the inner product on $T_x^*M$; let us denote this vector space by $V$.

Since $\vartheta(v + w)^2 = [\vartheta(v) + \vartheta(w)]^2$, it follows from (7) that

$$\vartheta(v)\vartheta(w) + \vartheta(w)\vartheta(v) = \langle v, w \rangle I, \quad \forall v, w \in V.$$ 

Hence, $\vartheta$ extends uniquely to an algebraic homomorphism

$$\vartheta : Cl(V, g) \to \text{End}(E).$$

This gives $E$ the structure of a module over $Cl(V, g)$ or a Clifford module.

If $E$ has a Hermitian metric and (6) also holds, i.e.

$$\vartheta(v) = \vartheta(v)^*, \quad \forall v \in V;$$

we call $E$ a Hermitian Clifford module.

- For this notion to be useful, we need the inner product to be positive-definite.
• We compute the symbols of $d$ and $\delta$. Since, for a $k$-form $u$,

$$d(ue^{i\lambda\psi}) = i\lambda e^{i\lambda\psi}(d\psi) \wedge u + e^{i\lambda\psi}du,$$

we see that

$$\frac{1}{i}\sigma_d(x, \xi) u = \xi \wedge u.$$

By (5), we have

$$\sigma_d(x, \psi) = \sigma_d(x, \xi)^* = -i\xi u.$$

Hence

$$-\sigma_d(x, \xi) u = (i\xi \xi) \wedge u + \xi \wedge (i\xi u).$$

If we perform the calculation by picking an orthonormal basis for $T^*_x M$ of the form $[e_1 \cdots e_n]$ with $\xi = |\xi|e_1$, we see that

$$\sigma_d(x, \xi) u = -|\xi|^2 u.$$

• On a vector space $V$ with a positive-definite inner product, if we define $E_v : \Lambda^j V \to \Lambda^{j+1} V$ by

$$E_v(v_1 \wedge \cdots \wedge v_j) = v \wedge v_1 \wedge \cdots \wedge v_j,$$

then $i_{\theta^*} : \Lambda^{j+1} V \to \Lambda^j V$ is its adjoint.

The principal symbol of $d + \delta$ on $V = T^*M$ is $\frac{1}{i}(E_v - i_{\theta^*})$.

That is to say,

$$(8) \quad iM(v) = E_v - i_{\theta^*}$$

defines a linear map from $V$ into $\text{End}(\Lambda^* V)$ which extends to an algebra homomorphism

$$M : Cl(V, g) \to \text{End}(\Lambda^* V).$$

The anticommutation relation

$$M(v)M(w) + M(w)M(v) = 2\langle v, w \rangle I$$

is equivalent to

$$(9) \quad E_v i_{\theta^*} + i_{\theta^*} E_v = \langle v, w \rangle I.$$
• The example just discussed gives rise to Hermitian Clifford module.
• We now show conversely that generally such module produce operators of Dirac type.
  – More precisely, if \( M \) is a Riemannian manifold, \( T^*_x M \) has an induced linear product, giving rise to a bundle
    \[ Cl(M) \to M \]
  or Clifford algebras. We suppose \( E \to M \) is a Hermitian vector bundle such that each fiber is a Hermitian \( Cl_x(M) \)-module (in a smooth fashion).
  – Let \( E \to M \) have a connection \( D \), so
    \[ D : C^\infty(M, E) \to C^\infty(M, T^*M \otimes E). \]
Now if \( E_x \) is a \( Cl_x(M) \)-module; the inclusion \( T^*_x M \hookrightarrow Cl_x(M) \) gives rise to a linear map
    \[ m : C^\infty(M, T^*M \otimes E) \to C^\infty(M, E) \]
called “Clifford multiplication”. We compose these two operators; set
\[
\mathcal{D} = \text{im} \circ D : C^\infty(M, E) \to C^\infty(M, E).
\]

• If \( U \) is an open subset of \( M \), on which we have an orthonormal frame \( \{e_i\} \) of smooth vector fields, with dual orthonormal frame \( \{v_j\} \) of 1-forms, then, for a section \( \varphi \) of \( E \),
\[
\mathcal{D}\varphi = i \sum v_j \cdot D_e^j \varphi \quad \text{on } U.
\]
Note that \( \sigma_{\mathcal{D}}(x,\xi)^* = \sigma_{\mathcal{D}}(x,\xi) \), so \( \mathcal{D} \) can be made symmetric by altering it at most by a zero-order term.

Lemma. \( \sigma_{\mathcal{D}}(x,\xi) = i\xi \).

Proof. Fix \( x \in M \) and an orthonormal basis \( e_1, \ldots, e_n \) of \( T_x M \).
  – Choose local coordinates \((x_1, \ldots, x_n)\) on \( M \) at \( x \) such that \( x \) corresponds to 0 and \( e_j \) corresponds to \((\partial/\partial x_j) \) \( \) for each \( j \).
  – Under the identification \( T^*_x M \cong T^*_x M \), \( e_j \) also corresponds to \((dx_j) \) for each \( j \).
  – For any local trivialization of \( E \) near \( x \), we have that
    \[
    D_e_j = \left( \frac{\partial}{\partial x_j} \right)_0 + \text{zero order terms}.
    \]
Hence, at 0, we have that
\[
\mathcal{D} = i \sum e_j \left( \frac{\partial}{\partial x_j} \right)_0 + \text{zero order terms}.
\]
Consequently, for any cotangle vector \( \xi = \sum \xi_j (dx_j)_0 \) at 0, we have by definition of the symbol that
\[
\sigma_{\mathcal{D}}(x,\xi) = i \sum e_j \xi_j = i\xi. \quad \Box
\]

Corollary. For \( v \in E_x \), \( \sigma_{\mathcal{D}}(x,\xi)v = m(\xi \otimes v) = \xi \cdot v \). Therefore \( \sigma_{\mathcal{D}}(x,\xi) \) is \(|\xi|_x\) times an isometry on \( E_x \), and \( \mathcal{D} \) is of Dirac type.
• Given a little more structure, we have more.

**Definition.** We say that a Levi-Civita connection $D$ on $M$ is a Clifford connection on $E$ if $D$ is also compatible with Clifford multiplication, in that

\[(12) \quad D_X(u \cdot \varphi) = (D_Xv) \cdot \varphi + v \cdot (D_X \varphi),\]

for a vector field $X$, a 1-form $v$, and a section $\varphi$ of $E$.

**Proposition.** If $D$ is a Clifford connection on $E$, then $D$ is symmetric.

**Proof.** Let $\varphi, \psi \in C^\infty(M, E)$. We want to show that

\[(13) \quad \int_M [(D\varphi, \psi) - (\varphi, D\psi)]dV = 0.\]

We can suppose $\varphi, \psi$ have compact support in a set $U$ on which local orthonormal frames $e_j, v_j$ as above are given.

- Define a vector field $X$ on $U$ by
  \[(X, v) = \langle \varphi, v \cdot \psi \rangle, \quad \forall v \in \Lambda^1 U.\]

**Claim:** $i \text{div } X = \langle D\varphi, \psi \rangle - \langle \varphi, D\psi \rangle$, pointwise in $U$; then (13) will follow from the divergence theorem. Indeed,

\[
\text{div } X = \sum \langle DX, v_j \rangle,
= \sum [e_j \cdot (X, v_j) - (X, De_j v_j)]
= \sum [e_j \cdot \langle \varphi, v_j \rangle - \langle \varphi, (De_j v_j) \cdot \psi \rangle]
= \sum [e_j \cdot (De_j \varphi, v_j \cdot \psi) - \langle \varphi, v_j \cdot De_j \psi \rangle], \quad \text{by (12).} \quad \square
\]

• If $E = E_0 \oplus E_1$ is a graded Hermitian $Cl(M)$-module, if $E_0$ and $E_1$ are each provided with metric connections, and if (12) holds, then the construction above gives an operator of Dirac type, of the form (3).

**Remark.** It is common to use Clifford algebras associated to negative-definite forms rather than positive-definite ones.

- The two types of algebras are simply related.
- If a linear map $\vartheta: V \to \text{End}(E)$ extends to an algebra homomorphism $Cl(V, g) \to \text{End}(E)$, then $i\vartheta$ extends to an algebra homomorphism $Cl(V, g) \to \text{End}(-E)$.
- If one uses a negative form, then the condition (6) that $E$ be a Hermitian Clifford module should be changed to
  \[
  \vartheta(v) = -\vartheta(v)^*, \quad \forall v \in V.
  \]

- In such a case, we should drop the factor of $i$ in (10) to associate the Dirac-type operator $D$ of $Cl(M)$-module $E$.
- In fact, getting rid of the factor of $i$ in (10) and (11) perhaps is the principal reason some people use the negative-quadratic form to construct Clifford algebras.