Superspaces

Definition. The superspace $E$ is $\mathbb{Z}_2$-graded vector space

$$E = E^+ \oplus E^-.$$  

A superalgebra is an algebra $A$ whose underlying vector space is a superspace, and whose products respects the $\mathbb{Z}_2$-grading; in other words,

$$A^i \cdot A^j \subset A^{i+j}.$$  

Example 1. An exterior algebra is a supertrace with the $\mathbb{Z}_2$-grading

$$\Lambda^\pm E = \sum_{(−1)^{i\pm1}} \Lambda^i E.$$  

Example 2. An ungraded vector space $E$ is implicitly $\mathbb{Z}_2$-graded with $E^+ = E$ and $E^- = 0$.

- The algebra of endomorphisms $\text{End}(E)$ of a superspace is a superalgebra, when graded in the usual way:

$$\text{End}^+(E) = \text{Hom}(E^+, E^+) \oplus \text{Hom}(E^-, E^-)$$

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Definition. A superbundle on a manifold $M$ is a bundle $E = E^+ \oplus E^-$, where $E^+$ and $E^-$ are two vector bundles on $M$ such that the fibers of $E$ are superspaces.

- If $E$ is a vector bundle, we will identify it with the superbundle $E^+ = E$ and $E^- = 0$.
- There is a general principle in dealing with elements of a superspace, that whenever a formula involves commuting an element $a$ past another $b$, one must insert a sign $(-1)^{|a|\cdot|b|}$, where $|a|$ is the parity of $a$, which equals 0 or 1 according to whether the degree of $a$ is even or odd.
- Thus, the supercommutator of a pair of odd parity elements of a superalgebra is actually their anticommutator, due to the extra minus sign.
- We will use the same bracket notation for this supercommutator as is usually used for the commutator:

$$[a, b] = ab - (-1)^{|a||b|}ba.$$

This bracket satisfies the axioms of a Lie superalgebra:

(i) $[a, b] + (-1)^{|a||b|}[b, a] = 0$

(ii)$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]].$
**Definition.** A superalgebra is said to be super-commutative if its superbracket vanishes identically.

**Example.** The exterior algebra $\Lambda E$ is super-commutative.

**Definition.** On a superalgebra $A$ is a linear form $\varphi$ on $A$ satisfying $\varphi([a, b]) = 0$.

- On a superalgebra $\text{End}(E)$, there is a canonical linear form given by the formula
  \[
  \text{Str}(a) = \begin{cases} 
  \text{Tr}_{E^+}(a) - \text{Tr}_{E^-}(a), & \text{if } a \text{ is even}, \\
  0, & \text{if } a \text{ is odd}.
  \end{cases}
  \]

**Proposition 1.** The linear form $\text{Str}$ defined above is a supertrace on $\text{End}(E)$.

**Proof.** We must verify that $\text{Str}[a, b] = 0$.

(i) If $a$ and $b$ have opposite parity, then $[a, b]$ is odd in parity and hence $\text{Str}[a, b] = 0$.

(ii) If $a = \begin{pmatrix} a^+ & 0 \\ 0 & a^- \end{pmatrix}$ and $b = \begin{pmatrix} b^+ & 0 \\ 0 & b^- \end{pmatrix}$ are both even, then

$$[a, b] = \begin{pmatrix} [a^+, b^+] & 0 \\ 0 & [a^-, b^-] \end{pmatrix}$$

has vanishing supertrace, since $\text{Tr}[a^+, b^+] = \text{Tr}[a^-, b^-] = 0$.

(iii) If $a = \begin{pmatrix} 0 & a^- \\ a^+ & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & b^- \\ b^+ & 0 \end{pmatrix}$ are both odd, then

$$[a, b] = \begin{pmatrix} a^- b^+ + b^- a^- & 0 \\ 0 & a^+ b^- + b^+ a^+ \end{pmatrix}$$

has supertrace

$$\text{Str}[a, b] = \text{Tr}_{E^+}(a^- b^+ + b^- a^-) - \text{Tr}_{E^+}(a^+ b^- + b^+ a^-) = 0.$$  \[ \square \]

- If $E = E^+ \oplus E^-$ and $F = F^+ \oplus F^-$ are two supertraces, then their tensor product $E \times F$ is the superalgebra with underlying vector space $E \otimes F$ and grading

$$\begin{align*}
(E \otimes F)^+ &= (E^+ \otimes F^+) \oplus (E^- \otimes F^-), \\
(E \otimes F)^- &= (E^+ \otimes F^-) \oplus (E^- \otimes F^+).
\end{align*}$$

- If $A$ and $B$ are superalgebras, then their tensor product $A \otimes B$ is the superalgebra whose underlying space is $\mathbb{Z}_2$-graded tensor product of $A$ and $B$, and whose product is defined by the rule

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{|b_1|\cdot|a_2|} a_1 a_2 \otimes b_1 b_2.$$
**Definition.** If $A$ is a supercommutative algebra and $E$ is a supertrace, we extend the supertrace on $\text{End}(E)$ to a map

$$\text{Str} : A \otimes \text{End}(E) \to A,$$

by the formula $\text{Str}(a \otimes M) = a \text{Str}(M)$ for $a \in A$ and $M \in \text{End}(M)$.

- The extension of the supertrace vanishes on supercommutators in $A \otimes \text{End}(M)$, since

$$[a \otimes M, b \otimes N] = (-1)^{|M||b|}ab \otimes [M, N]$$

where $A$ is supercommutative.

- If $E_1$ and $E_2$ are two supertraces, we may identify $\text{End}(E_1 \otimes E_2)$ with $\text{End}(E_1) \otimes \text{End}(E_2)$, the action being as follows:

$$(a_1 \otimes a_2)(e_1 \otimes e_2) = (-1)^{|a_2||e_1|}(a_1 e_1) \otimes (a_2 e_2),$$

where $a_i \in \text{End}(E_i)$ and $e_i \in E_i$.

**Definition.** If $E$ is a supertrace such that $\dim (E^\pm) = m_\pm$, we define its determinant line to be the one-dimensional vector space

$$\det(E) = (\Lambda^{m+}E^+)\otimes \Lambda^{m-}E^-.$$ 

- If $E^+ = E^-$, then $\det(E)$ is canonically isomorphic to $\mathbb{R}$.

**Definition.** If $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a superbundle, then its determinant line bundle $\det(\mathcal{E})$ is the bundle

$$\det(\mathcal{E}) = (\Lambda^{m+}\mathcal{E}^+)\otimes \Lambda^{m-}\mathcal{E}^-,$$

where $m_\pm = \text{rk}(\mathcal{E}^\pm)$.

- If $M$ is an $n$-dimensional manifold, $\det(TM)$ is the volume-form bundle $\Lambda^nT^*M$.

**Definition.** If $E$ is a Hermitian supertrace, we say that $u \in \text{End}^-(E)$ is odd self-adjoint if it has the form

$$u = \begin{pmatrix} 0 & u^- \\ u^+ & 0 \end{pmatrix},$$

where $u^+ : E^+ \to E^-$, and $u^-$ is the adjoint of $u^+$.

- Let $V$ be a real vector space with basis $e_i$ and dual basis $e^i$. Denote

$$\iota_k = \iota(e_k), \quad \varepsilon^k = \varepsilon(e^k),$$

where $\varepsilon^k$ is the exterior product and $\iota_k$ is the interior product.

- We will identify $\Lambda^kV^*$ with $(\Lambda^kV)^*$ by setting $\langle e^i, e_j \rangle = \delta^i_j$, where

$$I = \{1 \leq i^1 < \cdots < i^k \leq \dim V\}, \text{ and } J = \{1 \leq j_1 < \cdots < j_k \leq \dim V\}$$

are multiindices and $e^I = e^{i_1} \wedge \cdots \wedge e^{i_k}$, $e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}$.

- Using this identification, it is easy to see that $\varepsilon(\alpha)^* = \iota(\alpha) \in \text{End}(\Lambda V^*)$ and that $\iota(\nu)^* = \varepsilon(\nu) \in \text{End}(\Lambda V)$.

- If $A \in \text{End}(V)$, we denote by $\lambda(A)$ the unique derivation of the superalgebra $\Lambda V$ which coincides with $A$ on $V \subset \Lambda V$; it is given by the explicit formula

$$\lambda(A) = \sum_{j,k} (e^j, Ae_k)\varepsilon_j \iota^k.$$
Definition. A non-zero linear map \( T : \Lambda V \to \mathbb{R} \) which vanishes on \( \Lambda^k V \) for \( k < \dim(V) \) is called a Berezin integral.

- Let us explain why such a linear map is called an integral.
- If \( V \) is a real vector space, the superalgebra \( \Lambda V^* \) is the algebra of polynomial functions on the purely odd supertrace with
  \[ E^+ = 0, \quad E^- = V. \]

If \( e_i \) is a basis for \( V \) with dual basis \( e^i \), the element \( e^i \in \Lambda V^* \) play the role of coordinate function on \( V \).

- From this point of view, the interior multoplication \( \iota(\nu) : \Lambda V^* \to \Lambda V^* \), \( (\nu \in V) \),
  is the operation of differentiation in the direction \( \nu \).
- We see that a Berezin integral is an analogue of the Lebesgue integral:
  a Berezin integral \( T : \Lambda V^* \to \mathbb{R} \) is a linear form on the function space \( \Lambda V^* \) which vanishes on “partial derivatives”:
  \[ T \cdot \iota(\nu) \alpha = 0, \quad \forall \nu \in V \quad \text{and} \quad \alpha \in \Lambda V^*. \]

- If \( V \) is an oriented Euclidean vector space, there is a canonical Berezin integral, defined by projecting \( \alpha \in \Lambda V \) onto the component of the monomial \( e_1 \wedge \cdots \wedge e_n \); here \( n \) is the dimension of \( V \) and \( e_i \) form an oriented orthonormal basis of \( V \).
  - We will denote the Berezin integral by \( T \):
    \[ T(e_i) = \begin{cases} 1, & |f| = n, \\ 0, & \text{otherwise}. \end{cases} \]

If \( \alpha \in \Lambda V \), we will often denote \( T(\alpha) \) by \( \alpha_{[n]} \), although strictly speaking \( \alpha_{[n]} \) is an element of \( \Lambda^n V \) and not of \( \mathbb{R} \).

- If \( A \in \Lambda^2 V \), its exponential in the algebra \( \Lambda V \) will be denoted by \( \exp_A A = \sum (A^k / k!) \).

Definition. The Pfaffian of an element \( A \in \Lambda^2 V \) is the number
\[
Pf_A(A) = T(\exp_A A).
\]

The Pfaffian of an element \( A \in \text{Lie}(so(V)) \) is the number
\[
Pf(A) = T(\exp_A \sum_{i<j} \langle Ae_i, e_j \rangle e_i \wedge e_j).
\]

Example. If \( V = \mathbb{R}^2 \) with orthonormal basis \( \{e_1, e_2\} \), and if
\[
A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}
\]
then the Phaffian of \( A \) is \( \theta \).

- The Pfaffian vanishes if the dimension of \( V \) is odd.
- If the dimension of \( V \) is odd, \( Pf(A) \) is a polynomial of homogeneous order \( n/2 \) in the components of \( A \).
  - If the orientation of \( V \) is reversed, it changes sign.
**Proposition 2.** The Pfaffian of an antisymmetric linear map is a square root of the determinant

$$(\text{Pf}(A)^2) = \det A.$$ 

*Proof.* By the spectral theorem, we can choose an oriented basis $e_j$ of $V$ such that there are real numbers $c_j$, $1 \leq j \leq \frac{n}{2}$, for which

$$Ae_{2j-1} = c_j e_{2j},$$

$$Ae_{2j} = -c_j e_{2j-1}.$$ 

In this way we reduce the proof to the case in which $V$ is the vector space $\mathbb{R}^2$, and $Ae_1 = \theta e_2$, $Ae_2 = -\theta e_1$.

In this case $\text{Pf}(A) = \theta$, while the determinant of $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$ is $\theta^2$. □
Superconnections

- If $M$ is a manifold and $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ is a superbundle on $M$,
  - let $\mathcal{A}(M, \mathcal{E})$ be the space of $\mathcal{E}$-valued differential forms on $M$.
  - This space has a $\mathbb{Z}$-grading given by the degree of a differential form,
  - and we will denote the component of exterior degree $i$ of $\alpha \in \mathcal{A}(M, \mathcal{E})$ by $\alpha_{[i]}$.
  - In addition, we have the total $\mathbb{Z}_2$-grading, which we will denote by
    \[ \mathcal{A}(M, \mathcal{E}) = \mathcal{A}^+(M, \mathcal{E}) \oplus \mathcal{A}^-(M, \mathcal{E}) ; \]
    and which is defined by
    \[ \mathcal{A}(M, \mathcal{E}) = \mathcal{A}^{2i}(M, \mathcal{E}^i) \oplus \mathcal{A}^{2i+1}(M, \mathcal{E}^{i+1}) ; \]
    for example, the space of sections of $\mathcal{E}^\pm$ is contained in $\mathcal{A}^\pm(M, \mathcal{E})$.

- If $\mathfrak{g}$ is a bundle of Lie subalgebras on $M$, then $\mathcal{A}(M, \mathfrak{g})$ is a Lie superalgebra
  with respect to the Lie superbracket defined by
  \[ [\alpha_1 \otimes X_1, \alpha_2 \otimes X_2] = (-1)^{|X_1||\alpha_2|}(\alpha_1 \wedge \alpha_2) \otimes [X_1, X_2] . \]
  Likewise, if $\mathcal{E}$ is a superbundle of modulus for $\mathfrak{g}$ with respect to the action $\rho$,
  then $\mathcal{A}(M, \mathcal{E})$ is a supermodule for $\mathcal{A}(M, \mathfrak{g})$, with respect to the action
  \[ \rho(\alpha \otimes X)(\beta \otimes \nu) = (-1)^{|X||\beta|}(\alpha \wedge \beta) \otimes (\rho(X)\nu) . \]
  In particular, this construction may be applied to the bundle of Lie superalgebra $\text{End}(\mathcal{E})$, where $\mathcal{E}$ is a superbundle of $M$,
  since $\Lambda T^*M \otimes \mathcal{E}$ is a bundle of modules for the superalgebra $\Lambda T^*M \otimes \text{End}(\mathcal{E})$;
  we see that $\mathcal{A}(M, \text{End}\mathcal{E})$ is a Lie superalgebra, which has $\mathcal{A}(M, \mathcal{E})$ as a supermodule.

- Any differential operator on $\mathcal{A}(M, \mathcal{E})$ which supercommutes with the action of
  $\mathcal{A}(M)$ is given by the action of element of $\mathcal{A}(M, \text{End}\mathcal{E})$; such an operator will
  be called local.
  (This is consistent with the “super” point of view, that $\mathcal{A}(M)$ is the algebra of
  functions on a supermanifold fibered over $M$, and elements of $\mathcal{A}(M, \text{End}\mathcal{E})$ are
  the zeroth order differential operators on this supermanifold.)

Definition. If $\mathcal{E}$ is a bundle of supertraces over a manifold $M$, then a superconnection on $\mathcal{E}$ is an odd-parity first-order differential operator
  \[ \mathbb{A} : \mathcal{A}^\pm(M, \mathcal{E}) \rightarrow \mathcal{A}^\pm(M, \mathcal{E}) \]
  which satisfies Leibniz’s rule in the $\mathbb{Z}_2$-graded sense: if $\alpha \in \mathcal{A}(M)$ and $\theta \in \mathcal{A}(M, \mathcal{E})$,
  then
  \[ \mathbb{A}(\alpha \wedge \theta) = d\alpha \wedge \theta + (-1)^{|\alpha|}\alpha \wedge \mathbb{A}\theta . \]

- Let $\mathbb{A}$ be a superconnection on $\mathcal{E}$.
  The operator $\mathbb{A}$ can be extended to act on the space $\mathcal{A}(M, \text{End}\mathcal{E})$ in a way
  consistent with Leibniz’s rule:
    \[ \mathbb{A}\alpha = [\mathbb{A}, \alpha], \quad \forall \alpha \in \mathcal{A}(M, \text{End}\mathcal{E}) . \]
  To check that $\mathbb{A}\alpha$, as defined by this formula, is an element of $\mathcal{A}(M, \text{End}\mathcal{E})$, we
  need only check that the operator $[\mathbb{A}, \alpha]$ commutes with exterior multiplication
  by any differential form $\beta \in \mathcal{A}(M)$.
Definition. The curvature of a superconnection $\mathbb{A}$ is defined to be the operator $\mathbb{A}^2$ on $\mathcal{A}(M, \mathcal{E})$.

Proposition 3. The curvature is a local operator, and hence is given by the action of a differential form $F \in \mathcal{A}(M, \text{End}(\mathcal{E}))$, which has total degree even, and satisfies the Bianchi identity $\mathbb{A}F = 0$.

- A superconnection is entirely determined by its restriction to $\Gamma(M, \mathcal{E})$, which may be any operator $\mathbb{A} : \Gamma(M, \mathcal{E}^+) \to \mathcal{A}^-(M, \mathcal{E})$ that satisfies

$$\mathbb{A}(fs) = df \cdot s + f \mathbb{A}s, \quad \forall f \in C^\infty(M), \ s \in \Gamma(M, \mathcal{E}).$$

Indeed, if we define

$$\mathbb{A}(\alpha \otimes s) = d\alpha s + (-1)^{|\alpha|} \alpha \mathbb{A}s, \quad \forall \alpha \in \mathcal{A}(M), \ s \in \Gamma(M, \mathcal{E}).$$

this gives extension of $\mathbb{A}$ to $\mathcal{A}(M, \mathcal{E})$.

- In order to better understand what a superconnection consists of, we can break it into its homogeneous components $\mathbb{A}^{[i]}$, which maps $\Gamma(M, \mathcal{E})$ to $\mathcal{A}^i(M, \mathcal{E})$:

$$\mathbb{A} = \mathbb{A}^{[0]} + \mathbb{A}^{[1]} + \mathbb{A}^{[2]} + \cdots$$

Proposition 4. (1) The operator $\mathbb{A}^{[1]}$ is a covariant derivative on the bundle $\mathcal{E}$ which preserves the sub-bundles $\mathcal{E}^+$ and $\mathcal{E}^-$.

(2) The operator $\mathbb{A}^{[i]}$ for $i \neq 1$ are given by the action of differential forms

$$\omega^{[i]} \in \mathcal{A}^i(M, \text{End}(\mathcal{E}))$$

on $\mathcal{A}(M, \mathcal{E})$, where $\omega_i \in \begin{cases} \mathcal{A}^i(M, \text{End}^+(\mathcal{E})) & \text{if } i \text{ is even}, \\
\mathcal{A}^i(M, \text{End}^-(\mathcal{E})) & \text{if } i \text{ is odd.}
\end{cases}$

Corollary 5. The space of superconnections on $\mathcal{E}$ is an affine space modelled on the vector space $\mathcal{A}^-(M, \text{End}(\mathcal{E}))$.

Thus, if $\mathbb{A}_s$ is a smooth one-parameter family of superconnections, then

$$\frac{d\mathbb{A}_s}{ds} \in \mathcal{A}^-(M, \text{End}(\mathcal{E})) \ \forall s.$$

- We call $\mathbb{A}^{[1]}$ the covariant derivative component of the superconnection $\mathbb{A}$.

Example.