**Symmetric Spaces**

- A characteristic property of a symmetric space is that every point has a global symmetry that "reverses" the geodesics through that point.

**Definition.** *A Riemannian manifold $M$ is called* **locally symmetric** *if for every point $p \in M$ there exists a normal neighborhood $U$ of $p$ such that the map*

$$j_p = \exp_p \circ (-\mathrm{Id}) \circ \exp_p^{-1} : U \to M$$

*is an isometry. Here* $\mathrm{Id}_p$ *is the identity map on $T_pM$.*

- The map $j_p$ has the property "reversing" the geodesics that pass through the point $p$. This means that if $\gamma_v : (-\varepsilon, \varepsilon) \to U \subset M$ is the (unique) geometric with

$$\gamma_v(0) = p \quad \text{and} \quad \gamma_v'(o) = v \in T_pM,$$

  then
$$j_p(\gamma_v(t)) = \gamma_v(-t).$$
  Indeed, since $\gamma_v(t) = \exp_p(tv)$, we obtain that

$$j_p(\gamma_v(t)) = \exp \circ (-\mathrm{Id})(tv) = \exp_p(-tv) = \gamma_v(-t).$$

- For this reason the map $j_p$ is called a **locally geodesic symmetry** or simply a **local symmetry**.
- Forthermore, it is obvious that $j_P^2$ =Id, and if $v \in T_pM$, then

$$(dj_p)_p(v) = (dj_p)_p(\gamma_v'(0)) = (j_p \circ \gamma_v)'(0) = \gamma_{-v}'(0) = -v$$

  hence
$$(dj_p)_p = -\mathrm{Id}_p.$$
  Such an isometry is called an **involution**.

**Cartan's Theorem.** *A Riemannian manifold with curvature tensor $R$ is a locally symmetric space iff $DR = 0$.*

*Proof.* $(\Rightarrow)$ Since $j_p$ is an isometry, $dj_p$ commutes with $DR$. Hence

$$-(D_XR)(Y,Z)\omega = dj_p(D_XR)(Y,Z)(\omega) = D_{-X}R(-Y,-Z)(-\omega) = D_XR(Y,Z)\omega.$$

$(\Leftarrow)$ Note that we already have a conditate for a map; namely, if $\varepsilon$ is so small that $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p)$ is a diffeomorphism, then we can just define

$$j_p(x) = -x$$

in these coordinates. It remains to see **why this is an isometry when we have parallel curvature tensor**.

– To see this, we must show that **in these coordinates the metric is the same at $x$ and $-x$.**

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– Switching to geodesic polar coordinates, we have the usual equations relating curvature and the metric.
  So **by Lemma below**, the claim follows if we can **prove that the curvature tensor is the same when we go in opposite directions.**
– To check this, first observe

$$R(\cdot, v)v = R(\cdot, -v)(-v), \quad \forall v \in T_pM.,$$

  and so the curvature start out being the same.
  If $\partial_r$ is the radial field, we also have

$$(D_{\partial_r} R) = 0,$$

  and hence the curvature tensors not only start out being equal, but also satisfy the same first-order equation.
  Thus, they must remain the same as we go in opposite directions. $\square$

**Lemma.** *Let $M$, $\overline{M}$ be smooth manifolds with curvature tensors $R$, $\overline{R}$, respectively. Fix $p \in M$, $\overline{p} \in \overline{M}$ and let $I : T_pM \to T_{\overline{p}}\overline{M}$ be a linear isometry. Let $B_r(p)$ be a normal coordinate neighborhood of $p$. Define $\varphi : B_r(p) \to \overline{B}_r(p)$ by*

$$\varphi = \exp_{\overline{p}} \circ I \circ \exp_p^{-1}.$$

*Suppose that for all geodesics $\gamma$ emanating from $p$ we have*

$$I_\gamma(R(X, Y)Z) = \overline{R}(I_\gamma(X), I_\gamma(Y))I_\gamma(Z).$$

*Then $\varphi$ is an isometry and $d\varphi = I_\gamma$.*

*Proof of Lemma.* Given $X \in T_qM$, let $\gamma$ be the geodesic from $p$ to $q = \gamma(t^*)$ lying in $B_r(p)$, and let $J$ be the Jacobi field along $\gamma$ such that $J(0_p) = 0_p$ and $J(t^*) = X$. Let $\overline{\gamma} = \varphi(\gamma)$. Define $\overline{J}$ along $\overline{\gamma}$ by $\overline{J}(t) = I_\gamma(J(t))$.
– It follows immediately **from the hypothesis** that $\overline{J}(t)$ is a Jacobi field along $\overline{\gamma}$. Moreover,

$$\|J(t)\| = \|\overline{J}(t)\|.$$

⊙ To complete the proof it suffices to **claim:** $\overline{J}(t) = d\varphi(J(t))$
- Let $P_\gamma$ denote the parallel translation along $\gamma$. Then, from the relation

$$\overline{J}(t) = P_\gamma \circ I \circ P_{-\gamma}(J(t))$$

  it follws that

$$I(J'(0)) = \overline{J}'(0).$$

  Since $J$, $\overline{J}$ are Jacobi fields vanishing at $t = 0$, we have

$$J(t) = d\exp_{\gamma(0)}(tJ'(0)) \text{ and } \overline{J}(t) = d\exp_{\overline{\gamma}(0)}(t\overline{J}'(0)).$$

  Then

$$\begin{aligned}\overline{J}(t) =& d\exp_{\overline{\gamma}(0)} I(tJ'(0)) \\ =& d\exp_{\overline{\gamma}(0)} \circ dI \circ d\exp_{\gamma(0)}^{-1}(J(t)) = d\varphi(J(t)). \quad \square\end{aligned}$$

**Corollary.** *Manifolds of constant curvature are locally symmetric.*

**Definition.** *A connected Riemannian manifold $M$ is called a* **symmetric space** *if for each $p \in M$ there exists a (unique) isometry $i_p : M \to M$ such that*

$$i_p(p) = p \ \text{ and } \ (dj_p)_p = -\mathrm{Id}_p.$$

*The map $j_p$ is called a (global)* **symmetry** *of $M$ at $p$.*
*Equivalently, $\forall p \in M$, there exists an isometry $j_p : M \to M$ such that $j_p^2$ =Id, and $p$ is an isolated fixed point of $j_p$.*

**Examples 1.** The Euclidean space $\mathbb{R}^n$ is symmetric. The symmetry at $p \in \mathbb{R}^n$ is the map $j_p(x) = 2p - x$.

**Examples 2.** The sphere $\mathbb{S}^n$ is symmetric. Since its isometry group acts transitively on $\mathbb{S}^n$, it suffices to display a symmetry at the north pole $p = (1, 0, \cdots, 0)$, given by

$$j_p(x^1, \cdots, x^{n+1}) = (-x^1, \cdots, -x^n, x^{n+1}).$$

**The Structure of a Symmetric Space**

**Theorem 2.** *A symmetric Riemanniam manifold is geodesically complete.*

*Proof.* **: Claim: Every geodesic $\gamma(0, a) \to M$ is extendible**.
Let $b$ be near $a$ in (0,a) and let $j_{\gamma(b)}$ be the symmetry at $\gamma(b)$.
Since $j_{\gamma(b)}$ reverse geodesics through $\gamma(b)$, the required extension of $\gamma$ is $j_{\gamma(b)} \circ \gamma$. □

**Theorem 2.** *A symmetric Riemanniam manifold is homogeneous.*

*Proof.* **Claim:** $\forall p$, $q \in M$ there exists an isometry $\phi$ of $M$ that maps $p$ to $q$.
To prove this, note that since $M$ is geodesiclly complete by Theorem 1, any two points $p$, $q \in M$ can be joined by a geodesic.
– Let $\gamma : [0, 1] \to M$ be a geodesic with $\gamma(0) = q$, $\gamma(1) = p$.
– Then the symmetry $j_{\gamma(\frac{1}{2})}$ at the point $\gamma(\frac{1}{2})$ is an isometry; call it $\widetilde{\phi}$.
– This isometry reverses geodesics, hence carries $\gamma(1)$ to $\gamma(0)$. □

- Since $M$ is homogeneous, the isometry group $I(M)$ acts trsnsitively on $M$, and it can be shown that the identity component $G = I_o(M)$ of $I(M)$ also acts transitively.
  - By the Meyer-Steenrod Theorem, $I(M)$ is a Lie group.
  - Thus $M$ can be identified with the homogeneous space $G/K$, where $K$ is the isotropy subgroup of a point $p \in M$.
  - For simplicity take $p = eK = o$, and let $j$ denote the (global) symmetry of $M = G/K$ at $o$.
- Next, we will see that the symmetry $j$ provides $M$ with further structure.

**Definition.** *The map $\sigma : G \to G$ is defined by $\sigma(g) = j \circ g \circ j = j \circ g \circ j^{-1}$.*

Clearly, $\sigma(G)$ is a isometry, hence is an element of $I_o(M) = G$.
Thus $\sigma : G \to G$ is an automorphism.

**Definition.** *Let $G_\sigma = \{g \in G : \sigma(g) = g\}$ be the set of fixed point of $\sigma$, and $G_\sigma^o$ its connected component.*

**Theorem 3.** *(1) Let $M = G/K$ be a symmetric space with symmetry $j$ at $o = eK$. Then*

*(a) $\sigma^2 = \mathrm{Id}_G$, that is, $\sigma$ is an involution.*

*(b) The set $G_\sigma$ is a closed subgroup of $G$ such that $G_\sigma^o \subset K \subset G_\sigma$.*

*These two properties make the pair $(G, K)$ into what is called a symmetric pair.*

*(2) Conversely, if $G$ is a connected group, $K$ a closed subgroup of $G$, and $\sigma$ an automorphism of $G$ satisfying (a) and (b) above, then every $G$-invariant metric on $M = G/K$ makes $M$ into a Riemannian symmetric space such that*

$$j \circ \pi = \pi \circ \sigma.$$

*Here $j$ is the symmetry of $M$ at $o$, and $\pi : G \to M$ is the projection.*

- The map $\sigma : G \to G$ induces a map $\widetilde{\sigma} : G/K \to G/K$ which is called the **symmetry** of $G/K$.

**Proposition 4.** *Let $M = G/K$ be a symmetric space with involution $\sigma$, and Lie algebra $\mathfrak{g}$, $\mathfrak{k}$ of $G$ and $K$ respectively. Then*

(1) $\mathfrak{k} = \{X \in \mathfrak{g} : d\sigma(X) = X\}$.

(2) *If $\mathfrak{m} = \{X \in \mathfrak{g} : d\sigma(X) = -X\}$, then $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \mathfrak{k} \otimes \mathfrak{m}$.*

(3) *The subspace $\mathfrak{m}$ is $\mathrm{Ad}(K)$-invariant, that is, $\mathrm{Ad}(k)\mathfrak{m} \subset \mathfrak{m}$ for all $k \in K$. Hence, a symmetric space is reductive.*

(4) *The following are true:*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

*Proof.* (1) (i) First, we **claim: if $k \in K$, then $\sigma(k) = k$.**
Indeed, the differential of the isometry $\sigma(k)$ at $o$ is

$$dj_o \circ dk_o \circ dj_o = dk_o,$$

since $dj_o = -\mathrm{Id}_o$. Then the result is obtained from the general fact that *if two (local) isometries on a connected manifold have the same differential at a point, then they coincide.*

(i.a) Now let $X \in \mathfrak{k}$. Since as shown above, $\sigma\Big|_K = \mathrm{Id}_K$, we obtain that $d\sigma(X) = X$.

(ii) Conversely, let $X \in \mathfrak{g}$ with $d\sigma(X) = X$.
If $\alpha$ is the one-parameter subgroup that corresponds to $X$,
$\Rightarrow$ the curve $\sigma \circ \alpha$ is the one-parameter subgroup of $X$ with the same initial velocity,
$\Rightarrow$ $\sigma \circ \alpha = \alpha$.
$\Rightarrow$ $\alpha \in G_\sigma$, and in fact $\alpha \in G_\sigma^o \subset K$.
$\Rightarrow$ $X \in \mathfrak{k}$.

(2) (i) The sum is evidently direct. Now let $X \in \mathfrak{g}$ and set

$$X_{\mathfrak{k}} = \frac{1}{2}(X + d\sigma(X)), \quad X_{\mathfrak{m}} = \frac{1}{2}(X - d\sigma(X)).$$

Since $\sigma$ is an involution, so is $d\sigma$. Hence

$$d\sigma(X_{\mathfrak{k}}) = X_{\mathfrak{k}}, \quad d\sigma(X_{\mathfrak{m}}) = -X_{\mathfrak{m}},$$

which implies

$$X_{\mathfrak{k}} \in \mathfrak{k}, \quad X_{\mathfrak{m}} \in \mathfrak{m}.$$

Thus $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$.

(3) Let $X \in \mathfrak{m}$ and $k \in K$. In view of (2), we need to **claim:**

$$d\sigma(\mathrm{Ad}(k)X) = -\mathrm{Ad}(k)X.$$

Since $\sigma(k) = k$, the automorphisms $\sigma$ and the inner automorphism $I_k$ of $G$ commute; indeed,

$$\sigma I_k(g) = \sigma(kgk^{-1}) = \sigma(k)\sigma(g)\sigma(k^{-1}) = k\sigma(g)k^{-1} = I_k\sigma(g).$$

Thus we have that

$$\begin{aligned} d\sigma(\mathrm{Ad}(k)X) =& d(\sigma I_k)(X) = d(I_k\sigma)(X) \\ =& \mathrm{Ad}(k)d\sigma(X) = \mathrm{Ad}(k)(-X), \quad \text{by (2)}, \\ =& -\mathrm{Ad}(k)X. \end{aligned}$$

(4) The first inclusion holds since $K$ is a Lie subgroup of $G$.
The second is because of (3).
For the third, if $X, Y \in \mathfrak{m}$, then

$$d\sigma([X,Y]) = [d\sigma(X), d\sigma(Y)] = [-X,-Y] = [X,Y],$$

hence $[X,Y] \in \mathfrak{k}$. $\square$

**The Geometry of a Symmetric Space**

- As we saw in Proposition 4, a symmetric space is reductive, and we know that the $\mathrm{Ad}(K)$-invariant subspace $\mathfrak{m}$ can be naturally identified with the tangent space $T_o(G/K)$.
- Due to (4) in Proposition 4, $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$, **the natural reductive condition**

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle = \langle X, [Y,Z]_{\mathfrak{m}}\rangle \quad (X,Y,Z \in \mathfrak{m})$$

holds trivially; here $\langle\ ,\ \rangle$ is the scalar product on $\mathfrak{m}$, corresponding to the $G$-invariant metric of $G/K$. Thus we obtain the following.

**Proposition 5.** *Let $M = G/K$ be a symmetric space. Then the sectional curvature*

$$\langle R(X,Y)X,Y\rangle = \langle [[X,Y],X],Y\rangle, \quad \forall X,Y \in \mathfrak{m}.$$

*Proof.* Take into account that $M$ is naturally reductive (i.e. $U \equiv 0$), and the inclusion (4) of Proposition 4. $\square$