The Method of Characteristics for Quasilinear Equations

- Recall a simple fact from the theory of ODE’s:
  The equation $\frac{du}{dt} = f(t, u)$ can be solved (at least for small values of $t$) for each initial condition $u(0) = u_0$, provided that $f$ is continuous in $t$ and Lipschitz continuous in the variable $u$.

Recall that the solution may exist globally in time, or may blow up at some finite time.

- If we allow the equation and the initial condition to depend on a parameter $x$, then the solution $u$ depends on $x$ and may be written as $u(x, t)$.
  - In fact, $u$ becomes a solution of
    $$ \begin{cases} 
    u_t = f(x, t, u), \\
    u(x, 0) = u_0(x), 
    \end{cases} $$
    that may be thought of as an initial value problem for a PDE in which $u_x$ does not appear.
  - Assuming $f$ and $u_0$ are continuous functions of $x$, the solution $u(x, t)$ will be continuous in $x$ (and $t$).
  - Geometrically, the graph $z = u(x, t)$ is a surface in $\mathbb{R}^3$ that contains the curve $(x, 0, u_0(x))$.
    - This surface may be defined for all $t > 0$, or may blow up at some finite $t_0$ (which may depend on $x$).
    - However, if the surface remains bounded, then it will continue as a graph for all $t > 0$.
    - In particular, the surface cannot fold over on itself and thereby fail to be the graph of a function.

- These elementary ideas from ODE theory lie behind the method of characteristics which applies to general quasilinear first-order PDE’s, as we shall discover in this section.

Example (The Transport Equation). Consider the initial value problem for the transport equation
$$ \begin{cases} 
    u_t + au_x = 0, \\
    u(x, 0) = h(x), 
    \end{cases} $$
where $a$ is a constant.

Reduce this problem to an ODE along some curve $x(t)$ by finding $x(t)$ so that
$$ \frac{d}{dt} u(x(t), t) = au_x + u_t. $$

By the chain rule, we simply require $\frac{dx}{dt} = a$, i.e.,
$$ x = at + x_0, \text{ where } x_0 \text{ is the } x\text{-intercept of the curve.} $$

Along this curve we have $u_t = 0$, i.e. $u \equiv \text{constant} h(x_0) = h(x - at)$.

- Indeed, if $h$ is $C^1$, then we can check that $u(x, t) = h(x - at)$ satisfies the PDE and the initial condition.
- The lines $x = at + x_0$ are called the characteristic curves of $u_t + au_x = 0$.

- The reduction of a PDE to an ODE along its characteristics is called the method of characteristics, and applies to much more complicated equations.
- Let us now see how and why this method also applies to quasilinear PDEs.
1. Characteristics.

- We consider the quasilinear equation equations of the form
  \[ a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u), \]
  where \( u = u(x, y) \) is a continuously differentiable function of the two variables \( x \) and \( y \) and \( a, b, c \) are continuously differentiable functions of three variables \( x, y \) and \( u \).

- The solutions of (1) can be constructed via geometric arguments.
  - Namely, if \( u(x, y) \) is a solution of (1), let us consider its graph \( z = u(x, y) \).
  - The tangent plane to the graph of a solution \( u \) at a point \((x_0, y_0, z_0)\) has equation
    \[ u_x(x_0, y_0)(x - x_0) + u_y(x, y)(y - y_0) - (z - z_0) = 0 \]
  and the vector
    \[ n_0 = (u_x(x_0, y_0), u_y(x_0, y_0), -1) \]
  is normal to the plane. Introducing the vector
    \[ v_0 = (a(x_0, y_0, u), b(x_0, y_0, u), c(x_0, y_0, u)) \]
  then the equation (1) implies that
    \[ n_0 \cdot v_0 = 0. \]
  Thus, \( v_0 \) is tangent to the graph of \( u \), and thus must lie on the tangent plane to the graph of \( z = u(x, y) \) at the point \((x_0, y_0, z_0)\).
  - In other words, (1) says that
    \[ v(x, y, z) = (a(x, y, u), b(x, y, u), c(x, y, u)) \]
  defines a vector field in \( \mathbb{R}^3 \), to which graphs of solutions must be tangent at each point \((x, y, z)\).

**Definition.** Surfaces which are tangent at each point to a vector field in \( \mathbb{R}^3 \) are called integral surfaces of the vector field.
Curves which are tangent at each point to a vector field in \( \mathbb{R}^3 \) are called integral curves of the vector field.

- Thus to find a solution of (1), we should try to find integral surfaces of \( v \).

- How can we construct integral surfaces?
  We may construct integral surfaces of \( v \) as union of integral curves of \( v \), that is curves tangent to \( v \) at every point. These curves are solutions of the system
  \[ \frac{dx}{dt} = a(x, y, z), \quad \frac{dy}{dt} = b(x, y, z), \quad \frac{dz}{dt} = c(x, y, z) \]
  and are called characteristics. Note that \( z = z(t) \) gives the values of \( u \) along a characteristic; that is,
  \[ z(t) = u(x(t), y(t)). \]
  In fact, differentiating (3) and using (2) and (1), we have
  \[ c(x(t), y(t), z(t)) = \frac{dz}{dt} = u_x(x(t), y(t)) \frac{dx}{dt} + u_y(x(t), y(t)) \frac{dy}{dt} \]
  \[ = a(x(t), y(t), z(t))u_x(x(t), y(t)) + b(x(t), y(t), z(t))u_y(x(t), y(t)). \]
  Thus, along a characteristic the PDE (1) degenerates into an ODE.
The following proposition is a consequence of the above geometric reasoning and of the existence and uniqueness theorem for system of ODE’s.

**Proposition 1.** (a) Let the surface \( S \) be the graph of a \( C^1 \) function \( u = u(x, y) \). If \( S \) is a union of characteristics, then \( u \) is a solution of the equation (1); (in other words, a smooth union of characteristic curves is an integral surfaces).

(b) Every integral surface \( S \) of the vector field \( \mathbf{v} \) is a union of characteristics. Namely, every point of \( S \) belongs exactly to one characteristic, entirely contained in \( S \).

(c) Two integral surfaces intersecting at one point intersect along the whole characteristic passing through the point.

- In the case of a conservation law (with \( t = y \))
  \[ u_y + q'(u)u_x = 0 \quad \left( q'(u) = \frac{dq}{du} \right), \]
  we have introduced the notion of characteristics in a slightly different way, but we see below that there is no contradiction.

- **Conservation Laws.** According to the new definition, the characteristics of the equation
  \[ u_y + q'(u)u_x = 0 \quad \left( q'(u) = \frac{dq}{du} \right), \]
  with initial conditions
  \[ u(x, 0) = g(x) \]
  are the three-dimensional solution curves of the system
  \[ \frac{dx}{dt} = q'(z), \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 0 \]
  with initial conditions
  \[ x(s, 0) = s, \quad y(s, 0) = 0, \quad z(s, 0) = g(s), \quad s \in \mathbb{R}. \]
  Integrating, we find
  \[ z = g(s), \quad x = q'(g(s))t + s, \quad y = t. \]
  The projections of these straight lines on the \( (x, y) \)-plane are
  \[ x = q'(g(s))y + s, \]
  which are the “old characteristics”, called **projected characteristics** in the general quasilinear context.

- **Linear Equations.** Consider the linear equation
  \[ a(x, y)u_x + b(x, y)u_y = 0. \]
  Introducing the vector \( \mathbf{w} = (a, b) \), we may write this equation in the form
  \[ D_{\mathbf{w}}u = \nabla u \cdot \mathbf{w} = 0. \]
  Thus, every solution \( u \) is constant along the integral lines of the vector \( \mathbf{w} \), i.e. along the **projected characteristics**, which are solutions of the reduced characteristic system
  \[ \frac{dx}{dt} = a(x, y), \quad \frac{dy}{dt} = b(x, y), \]
  locally equivalent to the ODE \( b(x, y)dx - a(x, y)dy = 0. \)
The Cauchy Problem:

- Proposition 1 gives a characterization of the integral surfaces as a union of characteristics.
  - The problem is how to construct such unions to obtain a smooth surface.
  - One way to proceed is to proceed is to look for solutions \( u \) whose values are prescribed on a curve \( \gamma_0 \), contained in the \((x, y)\)-plane.

- In other words, suppose that \( x(s) = f(s), \quad y(s) = g(s), \quad s \in I \subset \mathbb{R} \)
is a parametrization of \( \gamma_0 \). We look for a solution \( u \) of (1) such that

\[
(4) \quad u(f(s), g(s)) = h(s), \quad s \in I,
\]

where \( h = h(s) \) is a given function.
We assume that \( I \) is a neighborhood of \( s = 0 \), and that \( f, g, h \) are in \( C^1(I) \).

- The system (1), (4) is called Cauchy problem.
  - If we consider the three-dimensional curve \( \Gamma_0 \) given by the parametrization

\[
x(s) = f(s), \quad y(s) = g(s), \quad z(s) = h(s),
\]

then, solving the Cauchy problem (1), (4) amounts to determining an integral surface containing \( \Gamma_0 \).

The Cauchy Problem: Given a curve \( \Gamma_0 \) in \( \mathbb{R}^3 \), can we find a solution \( u \) of (1) whose graph contains \( \Gamma_0 \)?

- In the special case that \( \Gamma_0 \) is the graph \((x, h(x))\) in the \(xz\)-plane of a function \( h \),
  the Cauchy problem is just an initial value problem with the obvious interpretation of the variable \( y \) as “time”.
- Namely, the data are assigned in the form of initial values

\[
u(x, 0) = h(x),
\]

with \( y \) playing the role of “time”. In this case, \( \gamma_0 \) is the axis \( y = 0 \) and \( x \) plays the role of the parameter \( s \).
Then a parametrization of \( \Gamma_0 \) is given by

\[
x = x, \quad y = 0, \quad z(x) = h(x).
\]

- By analogy, we often refer to \( \Gamma_0 \) as the initial curve.

- The strategy to solve a Cauchy problem comes from its geometric meaning:
since the graph of a solution \( u = u(x, y) \) is a smooth union of characteristics, we flow out from each point of \( \Gamma_0 \) along the characteristic curve through that point, thereby sweeping out an integral surface, which is the union of the characteristics and should give the graph of \( u \).
This is the method of characteristics.
• Actually, this construction of an integral surface containing $\Gamma_0$ can be achieved by writing $\Gamma_0$ as the graph of a curve

$$x(0) = f(s), \quad y(0) = g(s), \quad z(0) = h(s),$$

parameterized by $s \in I$, and then for each $s$ solving the system

$$\begin{align*}
\frac{dx}{dt} &= a(x, y, z), \\
\frac{dy}{dt} &= b(x, y, z), \\
\frac{dz}{dt} &= c(x, y, z)
\end{align*}$$

using (5) as initial conditions.

• Under our hypotheses, the Cauchy problem (5), (6) has a unique solution

$$x = X(s, t), \quad y = Y(s, t), \quad z = Z(s, t),$$

in a neighborhood of $t = 0$, for every $s \in I$.

- In this way, we obtain our integral surface parameterized by $s$ and $t$.
- To find the solution $u$ of (1), it remains only to replace the variables $s$ and $t$ by expressions involving $x$ and $y$.

• Thus, a couple of questions arise:

  (a) Do the three equations (7) define a function $z = u(x, y)$?
  (b) Even if the answer to (a) is positive, is $z = u(x, y)$ the unique solution of the Cauchy problem?

• Let us reason in a neighborhood of $s = t = 0$, setting

$$X(0, 0) = f(0) = x_0, \quad Y(0, 0) = g(0) = y_0, \quad Z(0, 0) = h(0) = z_0.$$

(a) The answer to question (a) is positive if we can solve for $s$ and $t$ in terms of $x$ and $y$ via the first two equations in (7), and find $s = S(x, y), t = T(x, y)$ of class $C^1$ in a neighborhood of $(x_0, y_0)$, such that

$$S(x_0, y_0) = 0, \quad T(x_0, y_0) = 0;$$

then, from the third equation $z = Z(s, t)$, we obtain

$$z = Z(S(x, y), T(x, y)) = u(x, y).$$

From the Inverse Function Theorem, the system

$$\begin{align*}
X(s, t) &= x \\
Y(s, t) &= y
\end{align*}$$

defines

$$s = S(x, y) \quad \text{and} \quad t = T(x, y)$$

in a neighborhood of $(x_0, y_0)$ if

$$J(0, 0) = \left| \begin{array}{cc} X_s(0, 0) & Y_s(0, 0) \\ X_t(0, 0) & Y_t(0, 0) \end{array} \right| \neq 0.$$
From (5) and (6), we have
\[
X_s(0, 0) = f'(0), \quad Y_s(0, 0) = g'(0)
\]
and
\[
X_t(0, 0) = a(x_0, y_0, z_0), \quad Y_t(0, 0) = b(x_0, y_0, z_0),
\]
so that (9) becomes
\[
J(0, 0) = \begin{vmatrix}
    f'(0) & g'(0) \\
    a(x_0, y_0, z_0) & b(x_0, y_0, z_0)
\end{vmatrix} \neq 0.
\]
or equivalently,
\[
b(x_0, y_0, z_0)f'(0) \neq a(x_0, y_0, z_0)g'(0).
\]
Geometrically, condition (11) means that the vectors
\[
(a(x_0, y_0, z_0), b(x_0, y_0, z_0)) \text{ and } (f'(0), g'(0))
\]
are not parallel: in other words, the tangent to \(\Gamma_0\) and the vector field \((a, b, c)\) along \(\Gamma_0\) project to vectors in the \(xy\)-plane which are nowhere parallel.

- **Conclusion:** if condition (10) holds, then (8) is a well-defined \(C^1\)-function in a neighborhood of \((x_0, y_0)\).

(b) The above consideration of \(u\) implies that the surface \(z = u(x, y)\) contains \(\Gamma_0\) and all the characteristics flowing out from \(\Gamma_0\), so that \(u\) is a solution of the Cauchy problem.

- Moreover, by Proposition 1 (c), two integral surfaces containing \(\Gamma_0\) must contain the same characteristics and therefore coincide.

- We summarize everything in the following, recalling that
\[
(x_0, y_0, z_0) = (f(0), g(0), h(0)).
\]

**Theorem 2.** Let \(a, b, c\) be \(C^1\)-functions in a neighborhood of \((x_0, y_0, z_0)\) and \(f, g, h\) be \(C^1\)-functions in \(I\). If
\[
J(0, 0) \neq 0,
\]
then, in a neighborhood of \((x_0, y_0)\), there exists a unique \(C^1\)-solution \(u = u(x, y)\) of the Cauchy problem
\[
\begin{cases}
    a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u) \\
    u(f(s), g(s)) = h(s).
\end{cases}
\]
Moreover, \(u\) is defined by the parametric equation (7).

**Remark.** If \(a, b, c\) and \(f, g, h\) are \(C^k\)-functions, \(k \geq 2\), then \(u\) is also a \(C^k\)-function.

- It remains to examine what happens when \(J(0, 0) = 0\), that is when the vectors \((a(x_0, y_0, z_0), b(x_0, y_0, z_0))\) and \((f'(0), g'(0))\) are parallel.
Suppose that there exists a $C^1$-solution $u$ of the Cauchy problem (12). Differentiating the second equation in (12), we obtain

$$h'(s) = u_x(f(s), g(s))f'(s) + u_y(f(s), g(s))g'(s).$$

Computing at $x = x_0$, $y = y_0$, $z = z_0$ and $s = 0$, we obtain

$$\left\{ \begin{array}{l}
a(x_0, y_0, u_0)u_x(x_0, y_0) + b(x_0, y_0, u_0)u_y(x_0, y_0) = c(x_0, y_0, u_0) \\
f'(0)u_x(x_0, y_0) + g'(0)u_y(x_0, y_0)g'(0) = h'(0) .
\end{array} \right.$$  

(14)

Since $u$ is a solution of the Cauchy problem, the vector $(u_x(x_0, y_0), u_y(x_0, y_0))$ is a solution of the algebraic system (14).

But then we know that, if $J(0, 0) = 0$, the condition

$$\text{rank} \left( \begin{array}{ccc}
a(x_0, y_0, z_0) & b(x_0, y_0, z_0) & c(x_0, y_0, z_0) \\
f'(0) & g'(0) & h'(0)
\end{array} \right) = 1$$

(15)

must hold; i.e., the two vectors

$$ (a(x_0, y_0, z_0), b(x_0, y_0, z_0), c(x_0, y_0, z_0)) \quad \text{and} \quad (f'(0), g'(0), h'(0)) \quad \text{must be parallel.}$$

This is equivalent to saying that $\Gamma_0$ is parallel to the characteristic curve at $(x_0, y_0, z_0)$.

When this occurs, we say that $\Gamma_0$ is characteristic at the point $(x_0, y_0, z_0)$.

**Conclusion.** If $J(0, 0) = 0$, a necessary condition for the existence of a $C^1$-solution $u = u(x, y)$ of the Cauchy problem in a neighborhood of $(x_0, y_0)$ is that $\Gamma_0$ be characteristic at $(x_0, y_0, z_0)$.

- Now assume that $\Gamma_0$ itself is a characteristic and let $P_0 = (x_0, y_0, z_0) \in \Gamma_0$.
  - If we choose a curve $\Gamma^*$ transversal to $\Gamma_0$ at $P_0$,
    by Theorem 2 there exists a unique integral surface containing $\Gamma^*$
    and, by Proposition 1 (c), this surface containing $\Gamma_0$.
  - In this way we can construct infinitely many solutions.

- We point out that the condition (15) is compatible with the existence of a $C^1$-solution only if $\Gamma_0$ is characteristic at $P_0$.
- On the other hand, it may occur that $\Gamma_0$ is noncharacteristic at $P_0$ and that the solutions of the Cauchy problem exist anyway; clearly, these solutions cannot be of class $C^1$.

- Let us summarize the steps to solve the Cauchy problem (12):

**Step 1.** Determine the solution (7) of the characteristic system (5) with initial conditions

$$X(s, 0) = f(s), \quad Y(s, 0) = g(s), \quad Z(s, 0) = h(s), \quad s \in I.$$
Step 2. Compute $J(s,t)$ on the initial curve $\Gamma_0$; i.e.

$$J(s,0) = \begin{vmatrix} f'(s) & g'(s) \\ X_t(s,0) & Y_t(s,0) \end{vmatrix}.$$

The following cases may occur:

Case 2a. $J(s,0) \neq 0$ for every $s \in I$; i.e. $\Gamma_0$ does not have characteristic points. Then, in a neighborhood of $\Gamma_0$, there exists a unique solution $u = u(x,y)$ of the Cauchy problem, defined by the parametric equation (7).

Case 2b. $J(s_0,0) = 0$ for some $s_0 \in I$ and $\Gamma_0$ is characteristic at the point $P_0 = (f(s_0), g(s_0), h(s_0))$. A $C^1$-solution may exist in a neighborhood of $P_0$ only if the rank condition (15) holds at $P_0$.

Case 2c. $J(s_0,0) = 0$ for some $s_0 \in I$ and $\Gamma_0$ is not characteristic at $P_0$. There are no $C^1$-solutions in a neighborhood of $P_0$. There may exist less regular solutions.

Case 2d. $\Gamma_0$ is a characteristic. Then there exists infinitely many $C^1$-solutions in a neighborhood of $\Gamma_0$.

Example 1. Consider the nonhomogeneous Burger equation

$$uu_x + u_y = 1$$

If $y$ is the time variable, then $u = u(x,y)$ represents a velocity field of a flux of particles along the $x$-axis. Equation (16) states that the acceleration of each particle is equal to 1. Assume

$$u(x,0) = h(x), \quad x \in \mathbb{R}.$$

The characteristics are solutions of the system

$$\frac{dx}{dt} = z, \quad \frac{dy}{dt} = 1, \quad \frac{dz}{dt} = 1$$

and the initial curve $\Gamma_0$ has the parametrization

$$x = f(s) = s, \quad y = g(s) = 0, \quad z = h(s), \quad s \in \mathbb{R}.$$

The characteristics flowing out from $\Gamma_0$ are

$$X(s,t) = s + \frac{t^2}{2} + th(s), \quad Y(s,t) = t, \quad Z(s,t) = t + h(s), \quad s \in \mathbb{R}.$$

Since

$$J(s,t) = \begin{vmatrix} 1 + th'(s) & 0 \\ t + h(s) & 1 \end{vmatrix} = 1 + th'(s),$$

we have $J(s,0) = 1$ and we are in Case 2a: in a neighborhood of $\Gamma_0$ there exists a unique $C^1$-solution.
If, for instance, \( h(s) = s \), we find the solution
\[
 u = y + \frac{2x - y^2}{2 + 2y}, \quad \forall x \in \mathbb{R}, \quad y \geq -1.
\]

Now consider the same equation with initial condition
\[
 u\left(\frac{y^2}{4}, y\right) = \frac{y}{2},
\]
assigning the values of \( u \) on the parabola \( x = \frac{y^2}{4} \). A parametrization of \( \Gamma_0 \) is given by
\[
x = s^2, \quad y = 2s, \quad z = s, \quad s \in \mathbb{R}.
\]

Solving the characteristic system with these initial conditions, we find
\[
(17) \quad X(s, t) = s^2 + ts + \frac{t^2}{2}, \quad Y(s, t) = 2s + t, \quad Z(s, t) = s + t, \quad s \in \mathbb{R}.
\]

Observe that \( \Gamma_0 \) does not have any characteristic point, since its tangent vector \((2s, 2, 1)\) is never parallel to the characteristic direction \((s, 1, 1)\). However
\[
J(s, t) = \left| \begin{array}{cc} 2s + t & 2 \\ s + t & 1 \end{array} \right| = -t,
\]

which vanishes for \( t = 0 \), i.e. exactly on \( \Gamma_0 \). We are in Case 2c.

Solving for \( s \) and \( t \), \( t \neq 0 \), in the first two equations (17), and substituting into the third one, we find
\[
u(x, y) = \frac{y}{2} \pm \sqrt{x - \frac{y^2}{4}}.
\]

We have found two solutions of the Cauchy problem, satisfying the differential equation in the region \( x > \frac{y^2}{4} \). However, these solutions are not smooth in a neighborhood of \( \Gamma_0 \), since on \( \Gamma_0 \) they are not differentiable.

If \( \Gamma \) satisfies (9), the solution may develop singularities away from \( \Gamma \) (i.e. for larger values of \( t \)).

Geometrically, this may be due to the integral surface folding over on itself at some point \((x_1, y_1)\).

In such a case, the solution experiences a “gradient catastrophe” (i.e. \( u_x \) becomes infinite) as \((x, y) \to (x_1, y_1)\), and therefore the solution \( u \) cannot be both single-valued and continuous.

**Example 2.** Consider the initial value problem
\[
u u_x + y u_y = x
\]
with
\[
u(x, 1) = 2x, \quad x \in \mathbb{R}.
\]
The characteristics are solutions of the system
\[ \frac{dx}{dt} = z, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = x \]
and the initial curve \( \Gamma_0 \) has the parametrization
\[ x = f(s) = s, \quad y = g(s) = 1, \quad z = 2s, \quad s \in \mathbb{R}. \]
We easily checked that (9) is satisfied: \( 1 \neq 0 \).
- Note that the characteristic equation for \( y \) happens to be decouple and may be integrated to obtain \( y = c(s)e^t \), and the initial condition then yields \( y = e^t \).
- The equations for \( x \) and \( z \) form a 2 \( \times \) 2 system, which may be solved by finding eigenvalues and eigenvectors, or we can simply observe that
\[ \frac{d(x + z)}{dt} = x + z \quad \text{and} \quad \frac{d(x - z)}{dt} = -(x - z), \]
which yields
\[ x + z = c_1(s)e^t \quad \text{and} \quad x - z = c_2(s)e^{-t}. \]
Using the initial conditions, we evaluate \( c_1 \) and \( c_2 \), then solve for \( x \) and \( z \):
\[ x = \frac{3}{2}se^t - \frac{1}{2}se^{-t}, \quad y = e^t, \quad z = \frac{3}{2}se^t + \frac{1}{2}se^{-t}. \]
Notice that \( z \) is defined for all \( s \) and \( t \), but if we eliminate \( s \) and \( t \) in favor of \( x \) and \( y \) we obtain our solution
\[ u(x, y) = x\frac{3y^2 + 1}{3y^2 - 1}, \]
which exists for \( |y| < \frac{1}{\sqrt{3}} \); a blow-up singularity has developed at \( y = \frac{1}{\sqrt{3}} \), which is where \( x_s y_t - y_s x_t \) vanishes.

- **Semilinear Equations.** Consider the semilinear equation
\[ (18) \quad a(x, y)u_x + b(x, y)u_y = c(x, y, u), \]
with initial curve \((f(s), g(s), h(s))\). The characteristic equations become
\[ \frac{dx}{dt} = a(x, y), \quad \frac{dy}{dt} = b(x, y), \quad \frac{dz}{dt} = c(x, y, z) \]
with the initial conditions
\[ x = f(s), \quad y = g(s), \quad z = h(s), \quad s \in \mathbb{R}. \]
The first two equations form a system, which may be solved to obtain a curve \((x(t), y(t))\) in the \( xy \)-plane, called **projected characteristics**. If we first find the projected characteristics, we can then integrate the third characteristic equation to find \( z \).
Moreover, regarding the problem of solving for \( s \) and \( t \) in terms of \( x \) and \( y \), the inverse function theorem tells us that this can be achieved provided the Jacobian matrix is nonsingular

\[
J \equiv \det \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} = x_s y_t - x_t y_s \neq 0.
\]

Notice that this condition is independent of the behavior of \( z \).

- In particular, at \( t = 0 \) we obtain the condition

\[
f'(s)b(f(s), g(s)) - g'(s)a(f(s), g(s)) \neq 0,
\]

which geometrically means that the projection of \( \Gamma \) into the \( xy \)-plane is a curve \( \gamma = (f(s), g(s)) \) that is nowhere parallel to the vector field \((a, b)\).

- (18) implies by continuity that (19) holds at least for small values of \( t \) so we have the following result.

**Proposition.** If the initial curve \( \gamma = (f(s), g(s)) \) satisfies (20), then there exists a unique solution \( u(x, y) \) of (18) in a neighborhood of \( \gamma \).

However, away from \( \gamma \) (i.e. for large values of \( t \)) the solution may develop a blow up type of singularity where \( J = 0 \);

in fact, even if (19) holds for all values of \( s \) and \( t \), the solution may develop a blow up type of singularity if the equation for \( \frac{dz}{dt} \) is nonlinear.

**Example 3.** Consider the initial value problem

\[
u_x + 2u_y = u^2
\]

with

\[u(x, 0) = h(x), \quad x \in \mathbb{R}.
\]

The characteristics are solutions of the system

\[
\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2, \quad \frac{dz}{dt} = z^2
\]

and the initial curve \( \Gamma_0 \) has the parametrization

\[x = f(s) = s, \quad y = g(s) = 0, \quad z = h(s), \quad s \in \mathbb{R}.
\]

We may integrate the first two equations (treating \( s \) as a constant) to find

\[x(s, t) = t + c_1(s), \quad y(s, t) = 2t + c_2(s),
\]

where the functions \( c_1(s) \) and \( c_2(s) \) may be determined from the initial conditions:

\[x(s, 0) = c_1(s) = s, \quad y(s, 0) = c_2(s) = 0,
\]

so that

\[x = t + s, \quad y = 2t.
\]
Notice that (19) holds for all $s$ and $t$, and we can explicitly solve for $s$ and $t$ to find

$$s = x - \frac{y}{2}, \quad t = \frac{y}{2}.$$

We may integrate the equation for $z$ to find $z(s, t) = -\frac{1}{1 + c_3(s)}$ and use the initial condition $z(s, 0) = h(s)$ to evaluate $c_3 = -1/h(s)$, so

$$z(s, t) = -\frac{h(s)}{1 - th(s)}.$$

Finally, we may eliminate $s$ and $t$ to express our solution as

$$u(x, y) = \frac{h(x - \frac{y}{2})}{1 - \frac{y}{2}h(x - \frac{y}{2})}.$$

– Notice that $u(x, 0) = h(x)$ and the solution $u(x, y)$ is certainly well defined for small enough values of $y$ (assuming $h$ is bounded); however, $u$ may become infinite if $y$ becomes large enough to cause the denominator to vanish.

– Even though the equations for $\frac{d}{dt}$ and $\frac{d}{dx}$ are linear and the solutions exist for all $s$ and $t$, the equations for $\frac{d}{dz}$ is nonlinear and may produce a singularity.

Example 4. Consider the PDE

(21) $u_x + xu_y = u^2$.

We find the projected characteristics by solving

$$\frac{dx}{1} = \frac{dy}{x}$$

to find the parabolas

$$y = \frac{1}{2}x^2 + C, \quad C: \text{constant}.$$

Take $\gamma$ to be the parabola $y = \frac{x^2}{2}$. Then for $\Gamma = (\gamma, z(\gamma))$ to be characteristic, we must have

$$\frac{dz}{z^2} = dx.$$

We may integrate this equation to find

$$-\frac{1}{z} = x + c, \quad c: \text{constant}.$$

The constant $c$ is determined by picking a point over $\gamma$ for $\Gamma$ to pass through.

– For example, if $\Gamma$ passes through $(0, 0, z_0)$, then $c = -\frac{1}{z_0}$, and $\Gamma$ is given by

$$z = -\frac{z_0}{1 - z_0x}.$$

For an arbitrary $C^1$ function $f$,

$$y - \frac{x^2}{2} = f(x + \frac{1}{z})$$

is a solution of (21).

– Along $\Gamma$, $y - \frac{x^2}{2} = 0$ and $x + \frac{1}{z} = \frac{1}{z_0}$, so that the solution will pass through $\Gamma$ provided $f(\frac{1}{z_0}) = 0$; with this sole restriction on $f$ we see that there is an infinite number of solutions passing through $\Gamma$. 
Remark. An interesting situation occurs when we want to find a solution of the linear equation

$$a(x, y)u_x + b(x, y)u_y = 0$$

in a smooth domain $\Omega \subset \mathbb{R}^2$, which assumes prescribed values on a subset of the boundary $\gamma = \partial \Omega$.

- For the solvability of the problem, we have to assign the Cauchy data only on the so called inflow boundary $\gamma_i$ defined by

$$\gamma_i = \{ \sigma \in \gamma : w \cdot \nu < 0 \},$$

where $w = (a, b)$ and $\nu$ is the unit outward normal to $\gamma$.

- If a smooth Cauchy data is prescribed on $\gamma_i$, a smooth solution is obtained by defining

(i) $u$ to be constant along each projected characteristic which meets the inflow boundary exactly once,

(ii) $u$ to be piecewise constant on those projected characteristics which is tangent to the inflow boundary at some point.

Observe that the points at which $w$ is tangent to $\gamma$ is characteristic.

Higher Dimensions

We now replace (1) with the equation

(22) $$\sum_{i=1}^{n} a_i(x_1, \cdots, x_n, u)u_{x_i} = c(x_1, \cdots, x_n, u).$$

The characteristic curves are now the integral curves of the system of $n+1$ equations of $n+1$ unknowns

$$\frac{dx_i}{dt} = a_i(x_1, \cdots, x_n, z), \quad \frac{dz}{dt} = c(x_1, \cdots, x_n, z).$$

As in the case $n = 2$, it is clear that if $u$ is a solution of (1), then the hypersurface

$$M_u = \{(x_1, \cdots, x_n, u(x_1, \cdots, x_n))\} \subset \mathbb{R}^{n+1}$$

is a union of characteristic curves.

- Now suppose we are given a bijection

$$\sigma = (\sigma_1, \cdots, \sigma_n) : \mathcal{D} \to \mathbb{R}^n, \quad \sigma_i = \sigma_i(s_1, \cdots, s_{n-1})$$

where $\mathcal{D} \subset \mathbb{R}^{n-1}$ is a compact $(n-1)$-dimensional manifold-with-boundary, and a function $h : \mathcal{D} \to \mathbb{R}$, $h = h(s_1, \cdots, s_{n-1})$.

- We can produce a solution $u$ of (22) with

$$u(\sigma(s)) = h(s), \quad \forall s \in \mathcal{D},$$

by taking the union of characteristic curves through all points $(\sigma(s), h(s)) \in \mathbb{R}^{n+1}$. 

Here we require that the matrix
\[
\begin{pmatrix}
D_1 \sigma_1(s) & \cdots & D_{n-1} \sigma_1(s) & a_1(\sigma(s), h(s)) \\
\vdots & \ddots & \vdots & \vdots \\
D_1 \sigma_n(s) & \cdots & D_{n-1} \sigma_n(s) & a_n(\sigma(s), h(s))
\end{pmatrix}
\]
be non-singular. This means that
(1) the matrix \((D_j \sigma_i(s))\) must have rank \(n - 1\), so that \(\sigma\) is an imbedding and \(\sigma(D) \subset \mathbb{R}^n\) is a hypersurface, and
(2) the vector \((a_1(\sigma(s), h(s)), \cdots, a_n(\sigma(s), h(s)))\) must lie in the tangent space of \(\sigma(D)\).

Thus the method of characteristics generates an \(n\)-dimensional integral manifold parameterized by \((s_1, \cdots, s_{n-1}, t)\). The solution \(u(x_1, \cdots, x_n)\) is obtained by solving for \((s_1, \cdots, s_{n-1}, t)\) in terms of the variables \((x_1, \cdots, x_n)\).

**Example 5.** Consider the initial value problem
\[
\begin{align*}
ux_1 + x_1u_x - u_x &= u \\
u(x_1, x_2, 1) &= x_1 + x_2, \quad x_1, x_2 \in \mathbb{R}.
\end{align*}
\]
The characteristics are solutions of the system
\[
\frac{dx_1}{dt} = 1, \quad \frac{dx_2}{dt} = x_1, \quad \frac{dx_3}{dt} = -1, \quad \frac{dz}{dt} = z.
\]
and the initial surface \(\Gamma_0\) is the hyperplane \(x_3 = 1, z = x_1 + x_2\), which is noncharacteristic and has the parametrization
\[
x_1 = s_1, \quad x_2 = s_2, \quad x_3 = 1, \quad z = s_1 + s_2.
\]
We find
\[
x_1 = t + s_1, \quad y = \frac{1}{2}t^2 + s_1 t + s_2, \quad x_3 = -t + 1, \quad z = (s_1 + s_2)e^t.
\]
We can then solve for \(s_1, s_2\) and \(t\) and plug into \(x\) to find
\[
u(x_1, x_2, x_3) = (x_1 + x_2 + (x_3 - 1)[1 + x_1 + \frac{1}{2}(x_3 - 1)])e^{1-x_3}.
\]
Notice that the solution exists for all values of \(x_1, x_2\) and \(x_3\).