General First Order Equations

- Let us write a general nonlinear equation $F(x, y, u, u_x, u_y) = 0$ as

$$F(x, y, u, p, q) = 0, \quad \text{where } p = u_x, q = u_y. \tag{1}$$

  We assume that $F = F(x, y, u, p, q)$ is a smooth function of its arguments and, to avoid trivial cases, that

$$F_p^2 + F_q^2 \neq 0.$$

– In the quasilinear case,

$$F(x, y, u, p, q) = a(x, y, u)p + b(x, y, u)q - c(x, y, u)$$

  and $F_p = a(x, y, u)$, $F_q = b(x, y, u)$ so that $F_p^2 + F_q^2 \neq 0$ says that a and b does not vanish simultaneously.

- As in the quasilinear case, we want to derive some ODEs which can integrate to find the characteristics.

Geometric Interpretation of (1):

– At each point $(x_0, y_0, z_0) \in \mathbb{R}^3$, we consider the set of all vectors $(p, q, -1)$ with

$$F(x_0, y_0, z_0, p, q) = 0,$$

  and the corresponding family $\mathcal{F}(x_0, y_0, z_0)$ of planes perpendicular to such vectors.

– If u is a solution of (1) and M_u is the graph of u, then the tangent plane of M_u at $(x_0, y_0, u(x_0, y_0))$ is a member of the family $\mathcal{F}(x_0, y_0, u(x_0, y_0))$.

– Now the relation

$$F(x_0, y_0, z_0, p, q) = 0,$$

  is one equation in the two unknowns p and q, so $\mathcal{F}(x_0, y_0, z_0)$ ought to be a one-parameter family of planes.

– In fact, if we assume that $F_q(x_0, y_0, z_0, p, q) \neq 0$, then the implicit function theorem determines q as a function of p:

$$F(x_0, y_0, z_0, p, q(p)) = 0, \quad \forall p.$$

  The possible tangent planes to the graph $z = u(x, y)$ are given by

$$z - z_0 = p(x - x_0) + q(p)(y - y_0), \tag{2}$$

  which, as p varies, describes a one-parameter family of planes through the point (x_0, y_0, z_0).

– This suggests that there is a cone $K(x_0, y_0, z_0)$, having it vertex at (x_0, y_0, z_0), with the property that a plane P is in $\mathcal{F}(x_0, y_0, z_0)$ iff P is tangent to $K(x_0, y_0, z_0)$ along a generator of this cone. Indeed, we recall the following definition.

Definition. *Suppose S_a is a one-parameter family of surfaces in $\mathbb{R}^3$ given by*

$$z = w(x, y; a); \tag{3}$$

w depends smoothly on x, y and the real parameter a. Consider also the equation

$$\partial_a w(x, y; a) = 0. \tag{4}$$

For a fixed value of a, these two equations (3), (4) determine a curve γ_a in $\mathbb{R}^3$. The **envelope** *$\mathcal{E}$ of the family of the surfaces S_a is the union of these curves γ_a.*

- ⊙ γ_a is the curve on the surface (3) approached by the intersection curve of the surface (3) and the surface $z = w(x, y; a + \delta a)$ as $\delta a \to 0$.
- ⊙ Along γ_a the tangent planes to the surface (3) and the envelope coincide.
- ⊙ The equation for $\mathcal{E}$ is found simply by solving (4) for a as a function of x and y, $a = f(x, y)$ and then substituting into (3) to obtain

  $$z = w(x, y; f(x, y)).$$

  Moreover, along γ_a, a is constant and we have

  $$dz = w_x dx + w_y dy, \qquad 0 = w_{ax} dx + w_{ay} dy.$$

Example. Let S_a be the one-parameter family of two-dimensional spheres in $\mathbb{R}^3$ of radius 1 and center $(a, 0, 0)$. Then the envelop $\mathcal{E}$ of this family is the cylinder of radius 1 centered on the x-axis.

Monge Cones

- If we apply this to the family (2), where p is the parameter, we obtain the "envelope" which is s surface C called the **Monge cone**, which satisfies (2) and

  $$\begin{cases} dz = p dx + q dy, \\ 0 = dx + \frac{dq}{dp} dy. \end{cases} \tag{5}$$

  If we differentiate (1) with respect to p, we obtain

  $$F_p + F_q \frac{dq}{dp} = 0;$$

  combining this with the second equation in (5), we obtain

  $$\frac{dx}{F_p} = \frac{dy}{F_q}.$$

  In consideration of the first equation in (5), these equations can be written in parametric form as

  $$\frac{dx}{dt} = F_p(x, y, z, p, q), \qquad \frac{dy}{dt} = F_q(x, y, z, p, q), \tag{6}$$
  $$\frac{dz}{dt} = p\frac{dx}{dt} + q\frac{dy}{dt} = pF_p(x, y, z, p, q) + qF_q(x, y, z, p, q).$$

Remark. Notice that, in general, the Monge cone C will not be a right circular cone, but rather a ruled surface, everywhere containing a line of tangency with one of the planes defined by (2).

- We can now imagine a field of cones in $\mathbb{R}^3$; a C^1 function $u : \mathbb{R}^2 \to \mathbb{R}$ is a solution of equation (1) iff the corresponding surface $M_u = \{(x, y, u(x, y))\}$ is tangent to the Monge cone $K(x_0, y_0, u(x_0, y_0))$ at each point $(x_0, y_0, u(x_0, y_0))$.
- This gives a field of directions at each point of M_u, namely the direction which lies along a generator of the Monge cone at that point.
- Namely, consider a solution u of (1), and let

$$z_0 = u(x_0, y_0), \quad p_0 = u_x(x_0, y_0), \quad q_0 = u_y(x_0, y_0).$$

The tangent plane of M_u at (x_0, y_0, z_0) consists of points (x, y, z) satisfying

$$z - z_0 = p_0(x - x_0) + q_0(y - y_0).$$

Equations (6) shows that points (x, y, z) which are on both this tangent plane and the Monge cone $K(x_0, y_0, z_0)$ have to satisfy

$$\frac{x - x_0}{F_p} = \frac{y - y_0}{F_q} = \frac{z - z_0}{p_0 F_p + q_0 F_q}, \quad [F_p, F_q \text{ evaluated at } (x_0, y_0, z_0, p_0, q_0)].$$

Therefore, these points lie along the line through (x_0, y_0, z_0) with direction

$$(7) \qquad (F_p, F_q, p_0 F_p + q_0 F_q) \quad [F_p, F_q \text{ evaluated at } (x_0, y_0, z_0, p_0, q_0)].$$

- An integral surface is defined to be **a surface S that, at each point (x_0, y_0, z_0), has a tangent plane P which is also tangent to the Monge cone C.**
- The unique line of tangency between C and P determines a direction field on S. Integral curves of this field are again defined to be **characteristic curves**, although they depend on the choice of the tangent plane P, i.e. on the choice of p_0, since this determines $q_0 = q(p_0)$.
- Notice that, in the quasilinear case $ap + bq = c$, the cone C degenerates to a line, so we need **not** specify p_0.

Characteristic Strips

- The system (6) correspond to the characteristic system for the quasilinear equations, but **with two more unknown functions $p(t)$ and $q(t)$**;
  thus the system (6) is underdetermined, and we need equations for $\frac{dp}{dt}$ and $\frac{dq}{dt}$.
- Proceeding formally, from (1) we can write

$$\begin{aligned}\frac{dp}{dt} &= u_{xx}(x(t), y(t))\frac{dx}{dt} + u_{xy}(x(t), y(t))\frac{dy}{dt} \\ &= u_{xx}(x(t), y(t))F_p + u_{xy}(x(t), y(t))F_q.\end{aligned}$$

⊙ We have to **get rid of the second order derivatives**.
- Since u is a solution (1), the identity

$$F(x, y, u(x, y), u_x(x, y), u_y(x, y)) = 0$$

holds. Partial differentiation with respect to x yields, since $u_{xy} = u_{yx}$,

$$F_x + F_u u_x + F_p u_{xx} + F_q u_{xy} \equiv 0.$$

Computing along $x = x(t)$, $y = y(t)$, we obtain

$$u_{xx}(x(t), y(t))F_p + u_{xy}(x(t), y(t))F_q = -F_x - p(t)F_u.$$

Thus, we define for p the following differential equation:

$$\frac{dp}{dt} = -F_x(x, y, u, p, q) - pF_u(x, y, u, p, q).$$

Similarly, we find

$$\frac{dq}{dt} = -F_y(x, y, u, p, q) - qF_u(x, y, u, p, q).$$

In conclusion, we are led to the following **characteristic system** of five autonomous equations

$$\tag{8} \begin{cases} \frac{dx}{dt} = F_p, \quad \frac{dy}{dt} = F_q, \quad \frac{dz}{dt} = pF_p + qF_q \\ \frac{dp}{dt} = -F_x - pF_u, \quad \frac{dq}{dt} = -F_y - qF_u. \end{cases}$$

Observe that $F = F(x, y, u, p, q)$ is a **first integral** of (8). In fact

$$\tag{9} \begin{aligned} &\frac{d}{dt}F(x(t), y(t), u(t), p(t), q(t)) \\ =&F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_u\frac{dz}{dt} + F_p\frac{dp}{dt} + F_q\frac{dq}{dt} \\ =&F_xF_p + F_yF_q + F_u(pF_p + qF_q) + F_p(-F_x - pF_u) + F_q(-F_y - qF_q) \\ \equiv&0, \end{aligned}$$

and therefore, if $F(x(t_0), y(t_0), u(t_0), p(t_0), q(t_0)) = 0$ at some t_0, then

$$F(x(t), y(t), u(t), p(t), q(t)) \equiv 0, \quad \text{for all } t.$$

Thus the curve

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

still called a **characteristic curve**, is still contained in an integral surface, while

$$p = p(t), \quad q = q(t)$$

gives the normal vector at each point, and can be associated with a piece of the tangent plane.

- We regard a point $(x_0, y_0, z_0, p, q) \in \mathbb{R}^5$ as a plane passing through (x_0, y_0, z_0) and perpendicular to the vector $(p, q, -1)$.
  - A curve $C(t) = (c_1(t), c_2(t), c_3(t), c_4(t), c_5(t))$ in $\mathbb{R}^5$ is then regarded as a family of planes, the plane at time t passing through $\Gamma(t) = (c_1(t), c_2(t), c_3(t))$ and perpendicular to the vector $(c_4(t), c_5(t), -1)$.
  - We refer to this curve $\Gamma(t) = (c_1(t), c_2(t), c_3(t))$ as the **base curve** of $C(t)$.
  - An arbitrary curve $C(t) = (c_1(t), c_2(t), c_3(t), c_4(t), c_5(t))$ in $\mathbb{R}^5$ is called a **strip** if the tangent vector $\Gamma'(t)$ of the base curve $\Gamma(t)$ always lies in the plane determined by $C(t)$ at time t. This means that

$$\Gamma'(t) = (c_1'(t), c_2'(t), c_3'(t)) \ \text{ is perpendicular to } (c_4(t), c_5(t), -1).$$

  So $C(t)$ is a strip iff it satisfies the **strip condition**

$$\frac{dc_3(t)}{dt} = c_4(t)\frac{dc_1(t)}{dt} + c_5(t)\frac{dc_2(t)}{dt}.$$

- Notice that any solution of (8) is automatically a **strip**.
- ⊙ For this reason, a solution $(x(t), y(t), u(t), p(t), q(t))$ of (8) is called **characteristic strip**, because the specification of p and q gives infinitesimal pieces of the tangent planes along the curve $(x(t), y(t), z(t))$.
- If u is a solution of (1), then M_u, the graph of u, is the union of certain base curves of characteristic strips.
- Notice that in order to construct our integral surface S, we are really only interested in the *base curve* of the strip, namely the curve $(x(t), y(t), z(t))$, but to find it we also need to find the functions $p(t)$ and $q(t)$.

The Cauchy Problem

- As usual, the **Cauchy problem** consists in looking for a solution u of (1), assuming prescribed values on a given curve γ_0 in the (x, y)-plane.
  - If γ_0 has the parameterization

$$x = f(s), \quad y = g(s), \quad s \in I \subseteq \mathbb{R}$$

  it is desired that

$$u(f(s), g(s)) = h(s), \quad s \in I,$$

  where $h = h(s)$ is a given function.
  - We assume that $0 \in I$ and the f, g, h are smooth functions in I.
- Let Γ_0 be the **initial curve**, given by the parameterization

$$x = f(s), \quad y = g(s), \quad z = h(s);$$

  this only specify the "initial" points for x, y and z.
- To solve the Cauchy problem for (1), we must assume the initial curve Γ_0 is **noncharacteristic**, i.e. at each point of Γ_0, the Monge cone is not tangent to Γ_0.

- Even so, the Cauchy problem as it stands is unreasonable.
  Geometrically, this is because (1) only determines a cone along Γ_0,
  and we do not know which direction to flow along a characteristic;
  analytically, we have a system of five ODEs to solve,
  but we only give initial values for x, y and z.
- The way to resolve this is to specify along Γ_0 two functions ϕ and ψ to give initial conditions for p and q.

- Suppose that we have an arbitrary curve $\Sigma = (f(s), g(s), h(s), \varphi(s), \psi(s))$ in $\mathbb{R}^5$, with base curve σ, and that $F(\Sigma(s)) = 0$ for all s.
  - There is a unique solution of (8) through each point $\Sigma(s)$,
    and by (9), this solution is a characteristic strip.
  - We thus obtain a family of characteristic strips $C(t)$.
  - The union of the corresponding base curves $c(t)$ is a surface M_u, containing the base curve $\sigma(s)$.
  - **In it reasonable to suppose now that u is a solution of (1)?**
    The answer is **no**, for there is no hope unless Σ **also a strip**.
    When **this** condition is satisfied, then everything works out.
- In fact, we will prove the following.

Theorem 1. *Assume that F is C^2 in a domain $D \subseteq \mathbb{R}^5$, and $F_p^2 + F_q^2 \neq 0$.
If $\sigma = (f, g) : [a, b] \to \mathbb{R}^2$ is a given C^2 curve, $h(s) : [a, b] \to \mathbb{R}$ is a given C^2 function, and $\varphi, \psi : [a, b] \to \mathbb{R}$ are two C^1 functions satisfying*

$$F(\Sigma(s)) = F(f(s), g(s), h(s), \varphi(s), \psi(s)) = 0, \quad \forall s \in [a, b], \tag{10}$$

and the **strip condition**

$$\frac{dh(s)}{ds} = \varphi(s)\frac{df(s)}{ds} + \psi(s)\frac{dg(s)}{ds}, \quad \forall s \in [a, b]. \tag{11}$$

If, furthermore, $\sigma'(s) = (f'(s), g'(s))$ and the projection of the characteristic direction (7) onto xy-plane namely, $F_p(\Sigma(s), F_q(\Sigma(s)))$, are linearly independent, i.e.

$$f'(s)F_q(\Sigma(s)) \neq g'(s)F_p(\Sigma(s)), \quad \forall s \in [a, b], \tag{12}$$

then there is a unique solution u of (1) satisfying

$$u(\sigma(s)) = h(s), \quad u_x(\sigma(s)) = \varphi(s), \quad u_y(\sigma(s)) = \psi(s), \quad \forall s \in [a, b],$$

Remark. Observe that if $(x_0, y_0, z_0) = (f(0), g(0), h(0))$ and p_0, q_0 is a solution of the system

$$\begin{cases} F(x_0, y_0, z_0, p_0, q_0) \equiv 0 \\ h'(0) = p_0 f'(0) + q_0 g'(0), \end{cases}$$

by the implicit function theorem, the condition

$$\begin{vmatrix} f'(0) & F_p(x_0, y_0, z_0, p_0, q_0) \\ g'(0) & F_q(x_0, y_0, z_0, p_0, q_0) \end{vmatrix} \neq 0$$

assures the existence of a solution $\varphi(s)$ and $\psi(s)$ of

$$\begin{cases} F(f(s), g(s), h(s), \varphi(s), \psi(s)) \equiv 0 \\ h'(s) = \varphi(s) f'(s) + \psi(s) g'(s). \end{cases}$$

in a neighborhood of (x_0, y_0).

Proof of Theorem 1. **(i)** Solve the characteristic system (8) with initial conditions

$$x(0) = f(s), \quad y(0) = g(s), \quad z(0) = h(s), \quad p(0) = \varphi(s), \quad q(0) = \psi(s).$$

Suppose we find the solution

$$x = X(s,t), \quad y = Y(s,t), \quad z = Z(s,t), \quad p = P(s,t), \quad q = Q(s,t) \tag{13}$$

of (8). If we can solve for s and t in terms of x and y via the first two equations in (13), and find $s = S(x,y)$, $t = T(x,y)$ of class C^1 in a neighborhood of (x_0, y_0), such that

$$S(x_0, y_0) = 0, \quad T(x_0, y_0) = 0;$$

then, from the third equation $z = Z(s,t)$ in (13), we obtain

$$z = Z(S(x,y), T(x,y)).$$

Let us reason in a neighborhood of $s = t = 0$, setting

$$X(0,0) = f(0) = x_0, \quad Y(0,0) = g(0) = y_0, \quad Z(0,0) = h(0) = z_0.$$

From the **Inverse Function Theorem**, the system

$$X(s,t) = x \quad \text{and} \quad Y(s,t) = y$$

defines

$$s = S(x,y) \quad \text{and} \quad t = T(x,y)$$

in a neighborhood of (x_0, y_0) if

$$0 \neq J(0,0) = \begin{vmatrix} X_s(0,0) & X_t(0,0) \\ Y_s(0,0) & Y_t(0,0) \end{vmatrix} = \begin{vmatrix} f'(0) & F_p(x_0, y_0, z_0, p_0, q_0) \\ g'(0) & F_q(x_0, y_0, z_0, p_0, q_0) \end{vmatrix}.$$

(ii) Define $u(x,y) = Z(S(x,y), T(x,y))$. We **claim:** in (13),

$$u_x(X(s,t), Y(s,t)) = P(s,t), \quad u_y(X(s,t), Y(s,t)) = Q(s,t). \tag{14}$$

This, (9) and (10) will prove

$$\begin{aligned} &F(x(t), y(t), u(x(t), y(t)), u_x(x(t), y(t)), u_y(x(t), y(t))) \\ &= F(x(0), y(0), u(x(0), y(0)), u_x(x(0), y(0)), u_y(x(0), y(0))) = 0. \end{aligned}$$

To prove the claim, we observe that we have, by (8),

$$\frac{\partial Z}{\partial t} = P\frac{\partial X}{\partial t} + Q\frac{\partial Y}{\partial t}. \tag{15}$$

Consider the function

$$\Lambda = \frac{\partial Z}{\partial s} - P\frac{\partial X}{\partial s} - Q\frac{\partial Y}{\partial s}$$

We will **claim**:

$$\Lambda(s,t) \equiv 0, \tag{16}$$

i.e.

$$\frac{\partial Z}{\partial s} = P\frac{\partial X}{\partial s} + Q\frac{\partial Y}{\partial s}. \tag{17}$$

Assume the truth of (16).

We observe that, differentiation the definition $u(x,y) = Z(S(x,y), T(x,y))$ gives

$$\text{(18)} \qquad \begin{cases} \frac{\partial Z}{\partial t} = u_x \frac{\partial X}{\partial t} + u_y \frac{\partial Y}{\partial t}, \\ \frac{\partial Z}{\partial s} = u_x \frac{\partial X}{\partial s} + u_y \frac{\partial Y}{\partial s}. \end{cases}$$

From (15) (17) and (18) we know that $(P(s,t), Q(s,t))$ and $(u_x(x,y), u_y(x,y))$ are solutions for a system of two linear equations in two unknowns, whose determinant

$$\det \begin{pmatrix} \frac{\partial X}{\partial t} & \frac{\partial Y}{\partial t} \\ \frac{\partial X}{\partial s} & \frac{\partial Y}{\partial t} \end{pmatrix} \neq 0,$$

in a neighborhood of $[a,b] \times \{0\}$. So the two solutions must be the same; in other words, (17) is true.

(iii) Thus it remains to **prove (16)**. We have, by (11),

$$\text{(19)} \qquad \Lambda(s,0) = \frac{\partial h(s)}{\partial s} - p(s)\frac{\partial f(s)}{\partial s} - q(s)\frac{\partial g(s)}{\partial s} = 0.$$

We **claim** that for each fixed s, $\Lambda(s,t)$ satisfies the ODE

$$\text{(20)} \qquad \frac{\partial \Lambda}{\partial t} = -F_u \Lambda,$$

which, together with the initial condition (18), implies (16).

– Thus, it remains to **prove (20)**. For this, letting

$$\beta(s,t) = (X(s,t), Y(s,t), Z(s,t), P(s,t), Q(s,t)),$$

we compute

$$\begin{aligned} \frac{\partial \Lambda}{\partial t} =& \frac{\partial^2 Z}{\partial s \partial t} - \frac{\partial P}{\partial t}\frac{\partial X}{\partial s} - \frac{\partial Q}{\partial t}\frac{\partial Y}{\partial s} - P\frac{\partial^2 X}{\partial s \partial t} - Q\frac{\partial^2 Y}{\partial s \partial t} \\ =& \frac{\partial}{\partial s}\left(\frac{\partial Z}{\partial t} - P\left(\frac{\partial X}{\partial t}\right) - Q\left(\frac{\partial Y}{\partial t}\right)\right) + \frac{\partial P}{\partial s}\frac{\partial X}{\partial t} + \frac{\partial Q}{\partial s}\frac{\partial Y}{\partial t} \\ & - \frac{\partial P}{\partial t}\frac{\partial X}{\partial s} - \frac{\partial Q}{\partial t}\frac{\partial Y}{\partial s} \\ =& 0 + F_p\left(\frac{\partial P}{\partial s}\right) + F_q\left(\frac{\partial Q}{\partial s}\right) + (F_x + F_u P)\frac{\partial X}{\partial s} + (F_y + T_u Q)\frac{\partial Y}{\partial s}, \\ & \qquad \text{(by (8), where all partial of } F \text{ are evaluated at } \beta(s,t)), \\ =& F_x\left(\frac{\partial X}{\partial s}\right) + F_y\left(\frac{\partial Y}{\partial s}\right) + F_z\left(\frac{\partial Z}{\partial s}\right) + F_p\left(\frac{\partial P}{\partial s}\right) + F_q\left(\frac{\partial Q}{\partial s}\right) \\ & - F_u\left(\frac{\partial Z}{\partial s} - P\frac{\partial X}{\partial s} - Q\frac{\partial Y}{\partial s}\right) \\ =& \frac{\partial}{\partial s}F(\beta(s,t)) - F_u \Lambda \\ =& -F_u \Lambda. \qquad \square \end{aligned}$$

- Thus, to solve the characteristic system, we have first to complete Γ_0 into a **strip** $\Sigma(s) = (f(s), g(s), h(s), \varphi(s), \psi(s))$.
  - The two functions $\varphi(s)$ and $\psi(s)$ represent the initial values for p and q and cannot be chosen arbitrarily.
  - A first condition is that each tangent plane must be tangent to the Monge cone; in other words, $\varphi(s)$ and $\psi(s)$ have to satisfy the equation (10) (recalling (9))
  - A second condition is that the planes must fit together smoothly along Γ_0 like scales of a fish, namely from differentiating $h(s) = u(f(s), g(s))$ we obtain the so called **strip condition** (11).
- Now we are in position to give a (formal) procedure to construct a solution of our Cauchy problem: **Determine a solution** $u = u(x, y)$ of $F(x, y, u, u_x, u_y) = 0$, **containing the initial curve** $(f(s), g(s), h(s))$:

Step 1: Solve for $\varphi(s)$ and $\psi(s)$ the (nonlinear) system

$$\begin{cases} F(f(s), g(s), h(s), \varphi(s), \psi(s)) \equiv 0 \\ h'(s) = \varphi(s)f'(s) + \psi(s)g'(s). \end{cases}$$

Step 2: Solve the characteristic system (8) with initial conditions

$$x(0) = f(s), \quad y(0) = g(s), \quad z(0) = h(s), \quad p(0) = \varphi(s), \quad q(0) = \psi(s).$$

Suppose we find the solution

$$x = X(s,t), \quad y = Y(s,t), \quad z = Z(s,t), \quad p = P(s,t), \quad q = Q(s,t).$$

Step 3: Solve

$$x = X(s,t), \quad y = Y(s,t)$$

for s, t in terms of x, y. Substitute

$$s = S(x,y), \quad t = T(x,y)$$

into $z = Z(s,t)$ to find a solution $z = u(x,y)$.

Example. Consider the initial value problem

$$\begin{cases} u = u_x^2 - 3u_y^2 \\ u(x,0) = x^2. \end{cases}$$

We have $F(p,q) = p^2 - 3q^2 - u$ and the characteristic system is

$$\frac{dx}{dt} = 2p, \quad \frac{dy}{dt} = -6q, \quad \frac{dz}{dt} = 2p^2 - 6q^2 = 2z \tag{21}$$

$$\frac{dp}{dt} = p, \quad \frac{dq}{dt} = q. \tag{22}$$

A parameterization of the initial line is

$$f(s) = s, \quad g(s) = 0, \quad h(s) = s^2.$$

To complete the initial strip we solve the system

$$\begin{cases} \varphi^2 - 3\psi^2 = s^2 \\ \varphi = 2s. \end{cases}$$

There are two solutions:

$$\varphi(s) = 2s, \quad \psi(s) = \pm s.$$

(i) The choice $\psi(s) = s$ yields, integrating equation (22),

$$P(s,t) = 2se^t, \quad Q(s,t) = se^t$$

whence, from (21),

$$X(s,t) = 4s(e^t - 1) + s, \qquad Y(s,t) = -6s(e^t - 1), \qquad Z(s,t) = s^2 e^{2t}.$$

Solving the first two equations for s, t and substituting into the third one, we obtain

$$u(x,y) = \left(x + \frac{2}{y}\right)^2.$$

(ii) The choice $\psi(s) = -s$ yields

$$u(x,y) = \left(x - \frac{2}{y}\right)^2.$$

- As the example shows, in general, there is **no uniqueness**, unless the system (10) has a unique solution.
- – On the other hand, if the system (10) has no (real) solution, then the Cauchy problem has no solution as well.

Higher Dimensions.

- Consider the general first-order PDE

(23) $\quad F(x_1, \cdots, x_n, u(x_1, \cdots, x_n), u_{x_1}(x_1, \cdots, x_n), \cdots, u_{x_n}(x_1, \cdots, x_n)) = 0.$

We denote the partials of F by F_{x_i}, F_u, F_{p_i}.

– Consider curves Γ in $\mathbb{R}^{n+1}$ satisfying

$$\begin{cases} \frac{dC_i(t)}{dt} = F_{p_i}(C(t)), & i = 1, \cdots, n \\ \frac{dC_{n+1}(t)}{dt} = \sum_{i=1}^n C_{n+1+i}(t) F_{p_i}(C(t)), & \\ \frac{dC_{n+1+i}(t)}{dt} = -F_{x_i}(C(t)) - C_{n+1+i}(t) F_u(C(t)), & i = 1, \cdots, n \end{cases}$$

As before, we easily check that if $C(t)$ satisfies this equations, then $F(C(t))$ is constant in t.

– A solution $C(t)$ with $F(C(t)) = 0$ for all t is called a **characteristic strip**.

⊙ Now suppose we are given a bijection

$$\sigma = (\sigma_1, \cdots, \sigma_n) : \mathcal{D} \to \mathbb{R}^n, \qquad \sigma_i = \sigma_i(s_1, \cdots, s_{n-1})$$

where $\mathcal{D} \subset \mathbb{R}^{n-1}$ is a compact $(n-1)$-dimensional manifold-with-boundary, and functions $h, \varphi_i, \cdots, \varphi_n : \mathcal{D} \to \mathbb{R}$, with

$$F(\Sigma(s)) = F(\sigma_1, \cdots, \sigma_n, h(s), \varphi_1(s), \cdots, \varphi_n(s)) = 0, \quad \forall s \in \mathcal{D}$$

Then there is a unique characteristic strip Γ through each point $\Sigma(s)$, and the union of the corresponding base curves is a hypersurface M_u.

– In order for the function u to be a solution to our PDE we will need two conditions, which allow us to extend the proof of Theorem 1 essentially without change.

(i) First the matrix

$$\begin{pmatrix} D_1\sigma_1(s) & \cdots & D_{n-1}\sigma_1(s) & F_{p_1}(\sigma(s), h(s)) \\ \vdots & \ddots & \vdots & \vdots \\ D_1\sigma_n(s) & \cdots & D_{n-1}\sigma_n(s) & F_{p_n}(\sigma(s), h(s)) \end{pmatrix}$$

must be non-singular. This means that

(1) the matrix $(D_j\sigma_i(s))$ must have rank $n-1$, so that σ is an imbedding and $\sigma(\mathcal{D}) \subset \mathbb{R}^n$ is a hypersurface, and

(2) the vector $(F_{p_1}(\Sigma(s)), \cdots, F_{p_n}(\Sigma(s))$ must not lie in the tangent space of $\sigma(\mathcal{D})$.

(ii) Second, we must have

$$\frac{\partial h}{\partial s_j} = \sum_{i=1}^{n} \varphi_i(s) \frac{\partial \sigma_i}{\partial s_j}$$

In terms of Σ, this condition reads

$$\frac{\partial \Sigma_{n+1}}{\partial s_j} = \sum_{i=1}^{n} \Sigma_{n+1+i}(t) \frac{\partial \Sigma_i}{\partial s_j},$$

and is called the **strip manifold condition**.

- ⊙ If we think of a point $(x_1, \cdots, x_n, z, p_1, \cdots, p_n)$ in $\mathbb{R}^{2n+1}$ as a hyperplane in $\mathbb{R}^{n+1}$ passing through $(x_1, \cdots, x_n, z)$ and perpendicular to the vector $(p_1, \cdots, p_n, -1)$, then $\Sigma : \mathcal{D} \to \mathbb{R}^{2n+1}$ may be regarded as a family of hyperplanes along the $(n-1)$-dimensional submanifold $\sigma(\mathcal{D})$.
- – It is easy to see that Σ satisfies the strip manifold condition iff the tangent space of $\sigma(\mathcal{D})$ at any point $\sigma(s)$ always lies in the hyperplane determined by Σ at s.

- We summerize by saying that we can solve the Cauchy problem for any strip manifold $\sigma_1, \cdots, \sigma_n, h(s), \varphi_1(s), \cdots, \varphi_n(s)$ for which the initial $(n-1)$-dimensional submanifold $\sigma(\mathcal{D})$ is noncharacteristic for the initial data $(h, \varphi_1, \cdots, \varphi_n)$.