Lax-Oleinik Formula.

Consider the initial-value problem for scalar conservation laws in one space dimension:

\[
\begin{align*}
\frac{u}{u_t} + (F(u))_x &= 0, \\
u(x, 0) &= g
\end{align*}
\tag{1}
\]

in \(\mathbb{R} \times (0, +\infty)\) on \(\mathbb{R} \times \{t = 0\}\).

The planar vector field \((u, -F(u))\) is curl-free and therefore, if \(u\) is smooth, there exists a potential \(w = w(x, t)\) such that

\[
w_x = u \quad \text{and} \quad w_t = -F(u).
\]

Thus, \(w\) is a solution of the Hamilton-Jacobi equation

\[
w_t + F(w_x) = 0.
\]

The initial data \(u(x, 0) = g(x)\) transforms into the initial data

\[
w(x, 0) = h(x) = \int_0^x g(y) \, dy \quad \forall x \in \mathbb{R}.
\]

If \(w\) is a smooth solution of the initial-value problem (2), (3) for the Hamilton-Jacobi equation, then \(u = w_x\) solves (1).

- Assume that the flux function \(F\) is uniformly convex.
  
  W.l.o.g. we may also take \(F(0) = 0\).

- Recall, however, that the Hopf-Lax formula for the solution of (2)

\[
w(x, t) = \min_{y \in \mathbb{R}} \left\{ t F^* \left( \frac{x - y}{t} \right) + h(y) \right\}, \quad \forall x \in \mathbb{R}, \ t > 0;
\]

we know that \(w\) is not in general smooth.

- But we know that \(w\) is differentiable a.e. Consequently,

\[
\frac{u}{u(x, t)} = \frac{\partial}{\partial x} \left[ \min_{y \in \mathbb{R}} \left\{ t F^* \left( \frac{x - y}{t} \right) + h(y) \right\} \right]
\]

is defined for a.e. \((x, t)\) and is presumably a leading candidate for some sort of weak solution of the initial-value problem (1).

- Our intention henceforth is to justify this expectation.

- First, we will need to rewrite the expression (4) into a more useful form.

Notation. Since \(F\) is uniformly convex, \(F'\) is strictly increasing and onto. Write

\[
G = (F')^{-1}
\]

for the inverse of \(F'\).
Theorem 1 (Lax-Oleinik Formula). Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex, and $g \in L^\infty(\mathbb{R})$.

(i) For each time $t > 0$, there exists for all but at most countably many values of $x \in \mathbb{R}$ a unique point $y(x, t)$ such that
\[
\min_{y \in \mathbb{R}} \left\{ tF^*(\frac{x-y}{t}) + h(y) \right\} = tF^*\left(\frac{x-y(x, t)}{t}\right) + h(y(x, t)).
\]

(ii) The mapping $x \mapsto y(x, t)$ is nondecreasing.

(iii) For each time $t > 0$, the function $u$ defined by (4) is
\[
u(x, t) = G\left(\frac{x-y(x, t)}{t}\right)
\]
In particular, formula (5) holds for a.e. $(x, t) \in \mathbb{R} \times (0, \infty)$.

Definition. Equation (6) is called the Lax-Oleinik formula for the solution of (1), where $h$ is defined by (3).

We first observe the following.

Lemma 2. Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex. Then $F^*$ is $C^2$. Furthermore,

\[(F^*)'(q) = G(q), \quad \forall q \in \mathbb{R}\]
Hence, we obtain the following.

Corollary. Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex. Then
\[(F^*)''(q) = G'(q) > 0, \quad \forall q \in \mathbb{R},\]
and consequently, $F^*$ is strictly convex.

Proof of Lemma 2. We note that
\[F^*(q) = \max_{p \in \mathbb{R}}(qp - F(p))
= qp^* - F(p^*),\]
where
\[F'(p^*) = q.\]
Then, by (5),
\[p^* = G(q),\]
and so
\[F^*(q) = qG(q) - F(G(q)).\]
In particular, $F^*$ is $C^2$. Furthermore,
\[(F^*)'(q) = G(q) + qG'(q) - F'(G(q))G'(q) = G(q), \quad \text{by (5).} \]
Lemma 3. Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex. Fix $t > 0$, $x_1 < x_2$. Suppose that $y_1 \in \mathbb{R}$ satisfies

\begin{equation}
\min_{y \in \mathbb{R}} \left\{ tF^*(\frac{x_1 - y}{t}) + h(y) \right\} = tF^*(\frac{x_1 - y_1}{t}) + h(y_1).
\end{equation}

Then

\begin{equation}
tF^*(\frac{x_2 - y_1}{t}) + h(y_1) < tF^*(\frac{x_2 - y}{t}) + h(y) \quad \text{if } y < y_1.
\end{equation}

Proof of Lemma 3. We have

\begin{equation}
\begin{cases}
x_2 - y_1 = \tau(x_1 - y_1) + (1 - \tau)(x_2 - y) \\
x_1 - y = (1 - \tau)(x_1 - y_1) + \tau(x_2 - y)
\end{cases}, \quad \text{for } 0 < \tau := \frac{y_1 - y}{x_2 - x_1 + y_1 - y} < 1.
\end{equation}

Since $(F^*)' > 0$, we thus have

\begin{equation}
\begin{cases}
F^*(\frac{x_2 - y_1}{t}) < \tau F^*(\frac{x_1 - y_1}{t}) + (1 - \tau)F^*(\frac{x_2 - y}{t}) \\
F^*(\frac{x_1 - y}{t}) < (1 - \tau)F^*(\frac{x_1 - y_1}{t}) + \tau F^*(\frac{x_2 - y}{t})
\end{cases},
\end{equation}

and hence

\begin{equation}
F^*(\frac{x_2 - y_1}{t}) + F^*(\frac{x_1 - y}{t}) < F^*(\frac{x_1 - y_1}{t}) + F^*(\frac{x_2 - y}{t}).
\end{equation}

Now notice from (8) that

\begin{equation}
tF^*(\frac{x_1 - y_1}{t}) + h(y_1) \leq tF^*(\frac{x_1 - y}{t}) + h(y).
\end{equation}

We multiply (10) by $t$, add $h(y_1) + h(y)$ to both sides, and subtract (11) from the resulting expression to obtain (9). \(\square\)

Proof of Theorem 1. Fix $t > 0$. We know that there exists at least one point $y_1 \in \mathbb{R}$ such that (8) is satisfied.

- Suppose $x_2 > x_1$. In view of (9), in computing the minimum of $tF^*(\frac{x_2 - y}{t}) + h(y)$, we need only consider those $y \geq y_1$, where $y_1$ satisfies (8).
- Now for each $x \in \mathbb{R}$ and $t > 0$, define the point $y(x, t)$ to be the smallest of those points $y$ giving the minimum of $tF^*(\frac{x - y}{t}) + h(y)$.
- Then the mapping $y(x, t)$ is nondecreasing and is thus continuous for all but at most countably many $x$.

• At a point $x$ of continuity of $y(\cdot, t)$, $y(x, t)$ is the unique value of $y$ which yields the minimum.

Step 3. We know that for each fixed $t > 0$, the mapping

$$x \mapsto w(x, t) := \min_{y \in \mathbb{R}} \left\{ tF^*(\frac{x - y}{t}) + h(y) \right\} = tF^*(\frac{x - y(x, t)}{t}) + h(y(x, t))$$

is differentiable a.e. Furthermore the mapping

$$x \mapsto y(x, t)$$

is monotone and consequently differentiable a.e. as well.
Thus given \( t > 0 \), the mappings
\[
x \mapsto F^* \left( \frac{x - y(x,t)}{t} \right) \quad \text{and} \quad x \mapsto h(y(x,t))
\]
are differentiable for a.e. \( x \).

Consequently the formula (4) becomes
\[
\frac{\partial}{\partial x} \left[ tF^* \left( \frac{x - y(x,t)}{t} \right) + h(y(x,t)) \right] = (F^*)' \left( \frac{x - y(x,t)}{t} \right) (1 - y_x(x,t)) + \frac{\partial}{\partial x} h(y(x,t)).
\]

But since \( y \mapsto tF^*(\frac{x-y}{t}) + h(y) \) has a minimum at \( y = y(x,t) \), the mapping \( z \mapsto tF^*(\frac{x-y}{t}) + h(y(z,t)) \) has a minimum at \( z = x \). Therefore,
\[
-(F^*)' \left( \frac{x - y(x,t)}{t} \right) y_x(x,t) + \frac{\partial}{\partial x} h(y(x,t)) = 0.
\]

And hence
\[
u(x,t) = (F^*)' \left( \frac{x - y(x,t)}{t} \right) = G \left( \frac{x - y(x,t)}{t} \right), \quad \text{by (7)}.
\]

**Theorem 2.** Under the assumptions of Theorem 1,
\[
u(x,t) = G \left( \frac{x - y(x,t)}{t} \right) = \frac{\partial}{\partial x} \left[ \min_{y \in \mathbb{R}} \left\{ tF^* \left( \frac{x-y}{t} \right) + h(y) \right\} \right]
\]
is a weak solution of (1), (2).

**Proof.** Define
\[
v(x,t) = \min_{y \in \mathbb{R}} \left\{ tF^* \left( \frac{x-y}{t} \right) + h(y) \right\},
\]
then \( v = u_x \). Recall that \( v \) is Lipschitz, differentiable a.e. and solves the Hamilton-Jacobi PDE
\[
\begin{cases}
v_t + F(v_x) = 0, & \text{in } \mathbb{R} \times (0, +\infty) \\
v = h & \text{on } \mathbb{R} \times \{t = 0\},
\end{cases}
\]
Choose any test function \( \phi \in C^\infty_0(\mathbb{R} \times (0, \infty)) \). Multiplying the PDE in (12) by \( \phi_x \) and integrating on \( \mathbb{R} \times (0, \infty) \), we obtain
\[
0 = \int_0^\infty \int_{-\infty}^{\infty} [v_t + F(v_x)] \phi_x \, dx dt.
\]
Thus, to show that \( u = v_x \) is a weak solution of (1), it suffices to claim:
\[
\int_0^\infty \int_{-\infty}^{\infty} v_t \phi_x \, dx dt = \int_0^\infty \int_{-\infty}^{\infty} v_x \phi_t \, dx dt + \int_{-\infty}^{\infty} g \phi \, dx \bigg|_{t=0}.
\]
To show this, we observe that
\[
\int_0^\infty \int_{-\infty}^{\infty} v_t \phi_x \, dx dt = - \int_0^\infty \int_{-\infty}^{\infty} v_x \phi_t \, dx dt - \int_{-\infty}^{\infty} v \phi_x \, dx \bigg|_{t=0}
\]
\[
= \int_0^\infty \int_{-\infty}^{\infty} v_x \phi_t \, dx dt + \int_{-\infty}^{\infty} v_x \phi \, dx \bigg|_{t=0}
\]
Now \( v(x,0) = h(x) = \int_{-\infty}^{x} g(y) \, dy \), and so \( v_x(x,0) = g(x) \) a.e. \( \square \)
Theorem 3. Under the assumptions of Theorem 1, there exists a constant $E$ such that the function $u$ defined by Lax's formula (4) satisfies

$$ u(x+z,t) - u(x,t) \leq \frac{E}{t}z $$

for all $t > 0$ and $x, z \in \mathbb{R}$ with $z > 0$.

Proof. Since $G = (F')^{-1}$, $G$ is nondecreasing. Thus, if $z > 0$,

$$ u(x,t) - u(x+z,t) = G\left(\frac{x-y(x,t)}{t}\right) - G\left(\frac{x-z-y(x+z,t)}{t}\right) \geq G\left(\frac{x-y(x+z,t)}{t}\right) - G\left(\frac{x-z-y(x+z,t)}{t}\right), $$

since $z > 0$ and $y(x,t)$ are nondecreasing. Recall that in computing the minimum in (4) we need only consider those $y$ such that $\frac{|x-y|}{t} \leq C$, for some constant $C$. Let $\tilde{G} = G\big|_{B_C(0)}$. Then $\tilde{G}$ is Lipschitz, and, by (14),

$$ u(x,t) - u(x+z,t) \geq -\frac{\text{Lip}(\tilde{G})z}{t}. \quad \Box $$. 

Asymptotics for Large Time

a. Decay at infinity

Theorem 4. Under the assumptions of Theorem 1, if we assume furthermore that $g$ is integrable, then there exists a constant $C$ such that the function $u$ defined by Lax's formula (4) satisfies

$$ |u(x,t)| \leq \frac{C}{t^{1/2}}, \quad \forall t > 0 \text{ and } x \in \mathbb{R}. $$

Proof. Setting

$$ \sigma \equiv F'(0) \quad \text{i.e.} \quad G(\sigma) = 0. \quad (15) $$

At the minimizing point $y(x,t)$ we have

$$ u(x,t) = G\left(\frac{x-y(x,t)}{t}\right), \quad \text{i.e.} \quad \left(\frac{x-y(x,t)}{t}\right) = F(u(x,t)). \quad (17) $$

The minimizing property of $y(x,t)$ yields

$$ tF^*\left(\frac{x-y(x,t)}{t}\right) + h(y(x,t)) \leq tF^*(\sigma) + h(x-\sigma t) = h(x-\sigma t); $$
that is,
\[ tF^*(\frac{x - y(x,t)}{t}) \leq \int_y^{x-\sigma t} g(z)dz. \]

Hence
\[ tF^*(\frac{x - y(x,t)}{t}) \leq \|g\|_{L^1}. \]

On the other hand, we obtain from (17)
\[ F^*(\frac{x - y(x,t)}{t}) = u(x,t)F'(u(x,t)) - F(u(x,t)) \]
\[ = (u(x,t)F'(u(x,t)) - F(u(x,t)) - F(0)). \]

Since
\[ 0 = F(0) = F(u(x,t)) + F'(u(x,t))(0 - u(x,t)) + \frac{1}{2}F''(\xi)(0 - u(x,t))^2, \]
where \( \xi \) is intermediate to 0 and \( u(x,t) \), we have
\[ F^*(\frac{x - y(x,t)}{t}) = \frac{1}{2}F''(\xi)(u(x,t))^2. \]

Substituting this into (19), we obtain
\[ u(x,t) \leq \sqrt{\frac{2\|g\|_{L^1}}{F''(\xi)t}}. \]
\[ \square \]

b. Decay to \( N \)-wave.

- Theorem 4 asserts that the \( L^\infty \)-norm of \( u \) goes to zero as \( t \to \infty \).
  - On the other hand we note from Example in the end that the \( L^1 \)-norm of \( u \) need not go to zero; indeed, the integral of \( u \) over \( \mathbb{R} \) is conserved.
  - We instead show here that \( u \) evolves in \( L^1 \) into a simple shape, assuming now that \( g \) has compact support.

**Definition.** Given constants \( p, q, d, \sigma \), with \( p, q \geq 0 \), \( d > 0 \), we define the corresponding \( N \)-wave to be the function
\[ N(x,t) = \begin{cases} \frac{1}{d}(\frac{x}{t} - \sigma) & \text{if } -(pd\xi)^{1/2} < x - \sigma t < (qd\xi)^{1/2} \\ 0 & \text{otherwise} \end{cases} \]
The constant \( \sigma \) is the velocity of the \( N \)-wave.

- Now set \( \sigma = F'(0) \) as in (15) and set
  \[ d := F''(0) > 0, \]
and also write
  \[ p = -2\min_{y \in \mathbb{R}} \int_{-\infty}^{y} g \ dx, \quad q = 2\max_{y \in \mathbb{R}} \int_{y}^{\infty} g \ dx. \]
Note that \( p, q \geq 0 \).
Theorem 5 (Asymptotics in $L^1$-norm). Assume that $p, q > 0$. Then there exists a constant $C$ such that

$$\int_{-\infty}^{\infty} |u(\cdot, t) - N(\cdot, t)| \, dx \leq \frac{C}{t^{1/2}}, \quad \forall t > 0.$$ 

To motivate this formulation, observe first that by Theorem 4, $|u| = O(t^{-1/2})$ and by definition $|N| = O(t^{-1/2})$. Thus we are able to formulate the following.

Lemma 6. For all $x$, there holds

$$|u(x, t) - N(x, t)| = O(t^{-1/2}), \quad as \ t \to \infty.$$ 

Moreover, we observe the following.

Lemma 7. There holds

$$\left| u(x, t) - \frac{1}{d} \left( x - \sigma t \right) \right| \leq \frac{|y(x, t)|}{t} + \frac{C}{t}, \quad \forall t > 0.$$

Proof of Lemma 7. By (16),

$$\frac{x - y(x, t)}{t} - \sigma = F'(u(x, t)) - F'(0) = F''(0)u(x, t) + O(|u(x, t)|^2).$$

Since $F''(0) = d$, we have

$$u(x, t) = \frac{1}{d} \left( x - \sigma t \right) - \frac{1}{d} \frac{y(x, t)}{t} + O(|u(x, t)|^2)$$

$$= \frac{1}{d} \frac{x - \sigma t}{t} - \frac{1}{d} \frac{y(x, t)}{t} + O(t^{-1}), \quad by \ Theorem \ 4. \ \square$$

• Since $g$ has compact support, the same is true for $u(\cdot, t)$ for each fixed $t > 0$. Set

$$s^+(t) = \inf \{ y : u(x, t) = 0, \forall x > y \}, \quad s^+(0) = s_+,$$

$$s^-(t) = \sup \{ y : u(x, t) = 0, \forall x < y \}, \quad s^-(0) = s_-.$$

Then $g(x) = 0$ if $x < s_-$ or $x > s_+$; thus,

$$u(x, t) \neq 0 \Rightarrow s_- \leq y(x, t) \leq s_+.$$
On the other hand, from (26) we obtain the following.

**Lemma 8.** If $s_- \leq y(x,t) \leq s_+$, then there holds

$$
|u(x,t) - \frac{1}{d} \frac{x - \sigma t}{t}| = O(t^{-1}), \text{ as } t \to \infty.
$$

To prove Theorem 5, in view of Lemma 6 and Lemma 8, it suffices to claim: there exist constants $A_1, A_2, B_1, B_2$ such that $A_2 \geq A_1, B_2 \geq B_1$,

$$
\begin{align*}
\text{(28)} \\
\left\{ s^+(t) - \sigma t \leq A_2 + (qdt)^{1/2}, \\
     s^-(t) - \sigma t \geq B_2 - (pdt)^{1/2},
\right.
\end{align*}
$$

and

$$
\begin{align*}
\text{(29)} \\
     s_- < y(x,t) < s_+ \quad \text{if } \quad B_2 - (pdt)^{1/2} \leq x - \sigma(t) \leq A_1 + (qdt)^{1/2}.
\end{align*}
$$

Indeed, the inequalities (28) and (29) imply

$$
\begin{align*}
\int_{-\infty}^{\infty} |u(x,t) - N(x,t)|dx
\leq \int_{A_2 + \sigma t + (qdt)^{1/2}}^{B_2 + \sigma t + (pdt)^{1/2}} |u(x,t) - N(x,t)|dx
+ \int_{B_2 + \sigma t - (pdt)^{1/2}}^{A_1 + \sigma t + (qdt)^{1/2}} |u(x,t) - N(x,t)|dx, \\
\text{for } t \text{ large enough,}
\leq O(t^{-1/2}) \cdot O(1) + O(t^{-1}) \cdot O(t^{1/2}), \quad \text{as } t \to \infty.
\end{align*}
$$

To establish (28), we may observe the following.

**Lemma 9.** There holds

$$
\dot{s}^+(t) \leq \sigma + \sqrt{\frac{dq}{t}} + O(t^{-1}), \quad \dot{s}^-(t) \geq \sigma + \sqrt{\frac{dp}{t}} + O(t^{-1}).
$$

From Lemma 9 we obtain (28).

**Corollary 10.** There holds

$$
\begin{align*}
     s^+(t) - \sigma t &\leq s_+ + \sqrt{qdt} + O(\ln t) \\
= s_+ + [\sqrt{dq} + O(t^{-1/2} \ln t)]t^{1/2} \\
     s^-(t) - \sigma t &\geq s_- - [\sqrt{dp} + O(t^{-1/2} \ln t)]t^{1/2}.
\end{align*}
$$
Proof of Lemma 9. Define
\[ u_\ell(t) = u(s^+(t) - 0, t), \]
and note that \( u(s^+(t) + 0, t) \equiv 0 \). It follows from the entropy condition that \( u_\ell(t) \geq 0 \) for every \( t > 0 \).
Moreover, since \( u_\ell(t) \geq 0 \) and \( F'(0) = \sigma \), the slope of the characteristic hitting the point \((s^+(t) - 0, t)\) in backward time is no less than \( \sigma \). This shows that
\[ \dot{s}^+(t) \geq \sigma. \]

- Suppose \( u_\ell(T) > 0 \) for some \( T > 0 \).
  Consider the two characteristics \( \Gamma_1, \Gamma_3 \) through the point \((s^+(T), T)\); along \( \Gamma_1 \), \( u \equiv u_\ell(T) \), while along \( \Gamma_3 \), \( u \equiv 0 \).
  Suppose that \( \Gamma_1 \) meets the \( x \)-axis at the point \((y, 0)\).
  The minimizing property of \( y \) and \( s^+(T) \) yield that equality holds in (18), i.e.,
\[ T(F'(\pi) - F(\pi)) = t^* F\left(\frac{s^+(t) - y}{t}\right) = \int_y^\infty g(x)dx. \]
This and (22) yield
\[ \frac{1}{2} F''(\xi)\pi^2 T = \int_y^\infty g(x)dx \leq \frac{q}{2}. \]
where \( \xi \) is intermediate to 0 and \( \pi \).
- Moreover, the Rankine-Hugoniot condition yields
\[ \dot{s}^+(t) = \frac{F(u_\ell(t))}{u_\ell(t)} = F'(0) + \frac{1}{2} F''(\theta) u_\ell(t) = \sigma + \frac{1}{2} F''(\theta) u_\ell(t), \]
where \( 0 < \theta < u_\ell(t) \). Hence from (31) we obtain
\[ \dot{s}^+(t) \leq \sigma + \frac{1}{2} \frac{F''(\theta)}{\sqrt{F''(\xi)}} \sqrt{\frac{q}{t}}. \]
But as \( t \to \infty \), it follows from Theorem 4 that both \( \theta \) and \( \xi \) tend to zero. Thus, since \( d = F''(0) \),
\[ F''(\theta) = d + O(|\theta|) \quad \text{and} \quad F''(\xi) = d + O(|\xi|). \]
Now from Theorem 4, we have
\[ \theta = O(|u|) = O(t^{-1/2}) \quad \text{and} \quad \xi = O(t^{-1/2}). \]
Thus
\[ \frac{F''(\theta)}{\sqrt{F''(\xi)}} = \sqrt{d} \frac{1 + O(t^{-1/2})}{\sqrt{1 + O(t^{-1/2})}} = \sqrt{d} (1 + O(t^{-1/2})). \]
It follows that
\[ \dot{s}^+(t) \leq \sigma + \frac{1}{2} \sqrt{\frac{dq}{t}} t^{-1/2} + O(t^{-1}). \]
The is the first inequalities in (30); the second one can be derived analogously. \( \Box \)
However, we still need to establish (29). For this, we observe by (16) that

\[ \frac{x - y(x,t)}{t} - \sigma = F'(u(x,t)) - F'(0) = F''(\theta)u(x,t), \]

where \( \theta \) is between \( u(x,t) \) and 0.

On the other hand, in view of (18), the minimizing property of \( y(x,t) \) gives

\[ tF^*(\frac{x - y(x,t)}{t}) \leq \int_y^{x - \sigma t} g(z)dz; \]

that is, in view of (20) and (21),

\[ \frac{t}{2} F''(\xi)(u(x,t))^2 \leq \frac{q}{2}, \quad \text{and} \quad \frac{t}{2} F''(\theta)(u(x,t))^2 \geq -\frac{p}{2} \]

from which we obtain

\[ -\sqrt{\frac{p}{F''(\xi)t}} \leq u(x,t) \leq \sqrt{\frac{q}{F''(\xi)t}}. \]

Substituting this into (33), we obtain

\[ -\frac{F''(\theta)}{\sqrt{F''(\xi)}} \sqrt{pt} \leq x - y(x,t) - \sigma t \leq \frac{F''(\theta)}{\sqrt{F''(\xi)}} \sqrt{qt}. \]

Using (32), we obtain

\[ -\sqrt{ptd(1 + O(t^{-1/2}))} \leq x - y(x,t) - \sigma t \leq \sqrt{qt}(1 + O(t^{-1/2})). \]

This yields

\[ y(x,t) - \sqrt{ptd}(1 + O(t^{-1/2})) \leq x - \sigma t \leq y(x,t) + \sqrt{qt}(1 + O(t^{-1/2})), \]

from which and (27) we obtain immediately a weaker version of Lemma 9.

**Lemma 10.** Set

\[ \varepsilon = \varepsilon(t) = \frac{A}{t^{1/2}}, \quad (t > 0), \quad A: \text{constant}. \]

If \( A \) is sufficiently large, then

\[ u(x,t) = 0 \quad \text{for} \quad x - \sigma t > s_x + (qd(1 + \varepsilon)t)^{1/2} \]

and

\[ u(x,t) = 0 \quad \text{for} \quad x - \sigma t < s_x - (pd(1 + \varepsilon)t)^{1/2} \]
Another immediate consequence of (34) is
\[
x - \sigma t - \sqrt{qd}(1 + O(t^{-1/2})) \leq y(x, t) \leq x - \sigma t + \sqrt{ptd}(1 + O(t^{-1/2}))
\]
and then the following.

**Lemma 11.** For \( A \) and \( t \) large enough,
\[
s_- < y(x, t) < s_+, \quad \text{if} \quad s_+ - (pd(1 - \varepsilon)t)^{1/2} < x - \sigma t < -s_- + (qd(1 - \varepsilon)t)^{1/2}.
\]

**Remark.** (35) implies
\[
((1 + \varepsilon)t)^{1/2} - t^{1/2} = O(1).
\]
Consequently Lemma 10 and Lemma 11 yield (29).

**Proof of Lemma 11.** Recall from the proof of Theorem 1 that the mapping \( x \mapsto y(x, t) \) is nondecreasing. Hence, too establish (34), it suffices to show that
\[
y(x, t) \leq s_+ \quad \text{if} \quad x - \sigma t = s_+ + (qd(1 - \varepsilon)t)^{1/2}
\]
\[
y(x, t) \geq s_- \quad \text{if} \quad x - \sigma t = s_+ - (pd(1 - \varepsilon)t)^{1/2},
\]
This, however, follows immediately from (38). \( \square \)

From Lemma 11 and Lemma 7, we obtain the following.

**Corollary 12.** For \( t \) large enough,
\[
\left| u(x, t) - \frac{1}{d} \left( \frac{x}{t} - \sigma \right) \right| \leq \frac{C}{t}, \quad \text{if} \quad s_- - (pd(1 - \varepsilon)t)^{1/2} < x - \sigma t < -s_- + (qd(1 - \varepsilon)t)^{1/2}.
\]

**Proof of Theorem 5.** For \( x \) large enough, we have \( s_+ + (qdt)^{1/2} > s_+ - (pdt)^{1/2} \), and thus
\[
\int_{-\infty}^{\infty} |u(x, t) - N(x, t)| \, dx = \int_{s_+ + (pd(1 + \varepsilon)t)^{1/2}}^{\sigma^{s_+ - (pd(1 + \varepsilon)t)^{1/2}}} |u(x, t) - N(x, t)| \, dx
\]
\[
\leq \int_{\sigma^{s_+ - (pd(1 - \varepsilon)t)^{1/2}}}^{\sigma^{s_+ + (qdt)^{1/2}}} |u(x, t) - N(x, t)| \, dx
\]
\[
+ \int_{\sigma^{s_+ - (pdt)^{1/2}}}^{\sigma^{s_+ - (pd(1 + \varepsilon)t)^{1/2}}} |u(x, t) - N(x, t)| \, dx
\]
\[
+ \int_{\sigma^{s_+ + (q(1 - \varepsilon)t)^{1/2}}}^{\sigma^{s_+ + (qdt)^{1/2}}} |u(x, t) - N(x, t)| \, dx
\]
\[
= O(t^{-1/2}) + O(1) + O(1) + O(1) + O(t^{1/2})
\]
\[
= O(t^{-1/2}), \quad \text{as} \quad t \to \infty. \quad \square
\]
Example. Consider the initial-value problem for Burger’s equation

\[
\begin{align*}
&u_t + (\frac{u^2}{2})_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\
&u = g \quad \text{on } \mathbb{R} \times \{t = 0\}
\end{align*}
\]

with the initial condition

\[
g(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 & \text{if } 0 \leq x \leq 1 \\
0 & \text{if } x > 1.
\end{cases}
\]

For \(0 \leq t \leq 2\), we find

\[
u(x, t) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{t}{t} & \text{if } 0 < x < t \\
1 & \text{if } t < x < 1 + \frac{t}{2} \\
0 & \text{if } x > 1 + \frac{t}{2}.
\end{cases}, \quad (0 \leq t \leq 2)
\]

For times \(t \geq 2\), we expect the shock wave parameterized by \(s(\cdot)\) to continue, with \(u = \frac{t}{t}\) to the left of \(s(\cdot)\), \(u = 0\) to the right.

- This is compatible with the entropy condition.
- We calculate the behavior of the shock curve by applying the Rankine-Hugoniot jump condition. Now

\[
[[u]] = \frac{s(t)}{t}, \quad [[F(u)]] = \frac{1}{2} \left(\frac{s(t)}{t}\right)^2, \quad \sigma = \dot{s}(t)
\]

along the shock curve for \(t \geq 0\). Thus

\[
\dot{s}(t) = \frac{s(t)}{2t} \quad (t \geq 2).
\]

Additionally \(s(2) = 2\), and so we can solve this ODE to find

\[
s(t) = (2t)^{1/2}, \quad t \geq 2.
\]

Hence we may augment (39) by setting

\[
u(x, t) = \begin{cases} 
0 & \text{if } x < 0 \\
\frac{t}{t} & \text{if } 0 < x < (2t)^{1/2} \\
0 & \text{if } x > (2t)^{1/2}
\end{cases}, \quad (t \geq 2).
\]