Hopf-Lax Formula

Lemma 1. Suppose that for each \( x \in \mathbb{R}^n \),
\[
\lim_{q \to \infty} \frac{L(x, q)}{|q|} = \infty.
\]
Then
\[
\sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} < \infty
\]
and \( \exists q^* \in \mathbb{R}^n \) such that
\[
\sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} = p \cdot q^* - L(x, q^*).
\]

Proof. Fixing \( p \), by (1), there exists a constant \( A \) such that
\[
L(x, q) \geq (|p| + 1)|q|, \quad \text{if } |q| \geq A.
\]
Consequently, if \( |q| \geq A \),
\[
p \cdot q - L(x, q) \leq -|q|
\]
and hence
\[
p \cdot q - L(x, q) \leq -L(x, 0) = p \cdot q - L(x, q)|_{q=0},
\]
if \( |q| \geq C \equiv \max(A, -L(x, 0)) \). Hence
\[
\sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} = \max_{q \in B_C(0)} \{ p \cdot q - L(x, q) \}.
\]
The continuity of \( L \) then yields the existence of \( q^* \). \( \square \)

Lemma 2. Suppose that \( L \in C^1 \) such that for each \( x \in \mathbb{R}^n \) (1) is satisfied and \( q \mapsto L(x, q) \) is convex
then the Legendre transformation of \( L \) is
\[
L^*(x, p) := \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} \quad (p \in \mathbb{R}^n).
\]

Proof. Suppose \( q^* \) is a maximum point of the mapping \( q \mapsto p \cdot q - L(x, q) \)
provided by Lemma 1. Then \( q^* \) being a critical point of this mapping, we have
\[
p = D_q L(x, q^*),
\]
and
\[
q^* = q(x, p).
\]
Thus
\[
\sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} = D_q L(x, q(x, p)) \cdot -L(x, q(x, p));
\]
the right hand side is exactly the definition of the Legendre transformation of \( L \). \( \square \)

• Thus, if \( L \) is convex in \( q \) and satisfies (1), we define the Legendre transform of \( L \) to be \( L^*(x, p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} \) and the Hamiltonian associated with \( L \) is
\[
H(x, p) = L^*(x, p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \}.
\]
This tells us how, under the convexity assumption and (1), to obtain \( H \) from \( L \).

• Now we ask the converse question: “given \( H \), how do we obtain \( L \)?”
**Theorem.** Suppose \( L \) is convex, satisfies (1) and \( H \) is defined by (3). Then for each \( x \in \mathbb{R}^n \),

(i) \( p \mapsto H(x, p) \) is convex, with
\[
\lim_{|p| \to \infty} \frac{H(x, p)}{|p|} = \infty.
\]

(ii) Furthermore,
\[
L(x, q) = H^*(x, q) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(x, p) \}.
\]

**Proof.** (i) For each fixed \( x \) and \( q \), the function \( p \mapsto p \cdot q - L(x, q) \) is linear, and so the mapping \( p \mapsto H(x, p) = L^*(x, p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} \) is convex. Moreover, fix \( \lambda > 0 \). Then
\[
H(x, p) = \sup_{q \in \mathbb{R}^n} \{ p \cdot q - L(x, q) \} \\
\geq \lambda |p| - L(x, \lambda \frac{p}{|p|}), \quad \text{choosing } q = \lambda \frac{p}{|p|},
\]
\[
\geq \lambda |p| - \max_{q \in \mathcal{B}_\lambda(0)} L(x, q).
\]

By the continuity of \( L \), \( \max_{q \in \mathcal{B}_\lambda(0)} L(x, q) \) is bounded. Hence, we obtain
\[
\liminf_{|p| \to \infty} \frac{H(x, p)}{|p|} \geq \lambda, \quad \forall \lambda > 0.
\]

(ii) In view of (3),
\[
H(x, p) + L(x, q) \geq p \cdot q, \quad \forall p, q \in \mathbb{R}^n,
\]
and consequently
\[
L(x, q) \geq \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(x, p) \} = H^*(x, q).
\]

On the other hand
\[
H^*(x, q) = L^{**}(x, p) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - L^*(x, p) \} \\
= \sup_{p \in \mathbb{R}^n} \{ p \cdot q - \sup_{r \in \mathbb{R}^n} \{ p \cdot r - L(x, r) \} \} \\
= \sup_{p \in \mathbb{R}^n} \inf_{r \in \mathbb{R}^n} \{ p \cdot (q - r) + L(x, r) \} \\
= \inf_{r \in \mathbb{R}^n} \sup_{p \in \mathbb{R}^n} \{ p \cdot (q - r) + L(x, r) \} \\
= \inf_{r \in \mathbb{R}^n} \{ p \cdot (q - r) + L(x, r) \}.
\]

Now since \( q \mapsto L(x, q) \) is convex, there exists \( s \in \mathbb{R}^n \) such that
\[
L(x, r) \geq L(x, q) + s \cdot (r - q) \quad \forall r \in \mathbb{R}^n;
\]
(if \( L \) is differentiable at \( q \), take \( s = D_qL(x, q) \)). Taking \( p = s \) in (5), we compute
\[
H^*(x, q) \geq \inf_{r \in \mathbb{R}^n} \{ s \cdot (q - r) + L(x, r) \} \geq L(x, q). \quad \square
\]
Hopf's formula

- Return to the Hamilton-Jacobi equation \( u_t + H(x, Du) = 0 \).
  Suppose \( H \) satisfies the following conditions for each \( x \in \mathbb{R}^n \):
  
  \( \text{(i)} \) \( p \mapsto H(x, p) \) is convex, and \( \text{(ii)} \) \( \lim_{|p| \to \infty} \frac{H(x, p)}{|p|} = \infty \).

  According to Theorem 1, \( L(x, q) = H^*(x, q) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(x, p) \} \) satisfies similar hypotheses.

- Recall now that the calculus of variations problem with Lagrangian \( L \) led to Hamilton's ODE for the associated Hamiltonian \( H \).
  Since these ODE are in turn the characteristic equations of the Hamilton-Jacobi PDE, we are led to conjecture that there is a direct connection between the PDE and the calculus of variations problem.

- So if \( x \in \mathbb{R}^n \) and \( t > 0 \) be given, we should presumably try to minimize the action

  \[
  \int_0^t L(w(s), \dot{w}(s)) ds
  \]

  over functions \( w : [0, t] \to \mathbb{R}^n \) satisfying \( w(t) = x \).

  But what should we take for \( w(0) \)?

  As we must somehow take into account the initial condition for our PDE, let us try to modify the action to include the function \( g \) evaluated at \( w(0) \):

  \[
  \int_0^t L(\dot{w}(s), w(s)) ds + g(w(0)).
  \]

  We follow up on this by guessing that a solution of (1), (2) should be given by the formula

  \[
  u(x, t) \equiv \inf \left\{ \int_0^t L(w(s), \dot{w}(s)) ds + g(y) \middle| w(0) = y, w(t) = x \right\},
  \]

  the infimum taken over all points \( y \in \mathbb{R}^n \) and all piecewise smooth functions \( w(s) \) with \( w(0) = y, w(t) = x \).

- To simplify further, let us suppose that \( H \), and thus \( L \), do not depend on \( x \). We are therefore assuming that

  \[
  H \text{ is convex and } \frac{H(p)}{|p|} \to \infty \text{ as } |p| \to \infty.
  \]

  Then

  \[
  L(q) = H^*(q) = \sup_{p \in \mathbb{R}^n} \{ p \cdot q - H(p) \}
  \]

  has the same properties. For this situation, formula (6) becomes

  \[
  u(x, t) \equiv \inf \left\{ \int_0^t L(\dot{w}(s)) ds + g(y) \middle| w(0) = y, w(t) = x \right\}.
  \]
• We propose now to investigate the sense in which $u$ so defined actually solves the Hamilton-Jacobi PDE

\[
\begin{cases}
  u_t + H(u_x) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\
  u = g & \text{on } \mathbb{R}^n \times \{t = 0\}.
\end{cases}
\]

We henceforth assume also that

\[
g : \mathbb{R}^n \to \mathbb{R} \text{ is Lipschitz;}
\]

this means that

\[
\text{Lip}(g) := \sup_{x, y \in \mathbb{R}^n \atop x \neq y} \left\{ \frac{g(x) - g(y)}{|x - y|} \right\} < \infty.
\]

• First we note that formula (8) can be further simplified:

**Theorem.** If $x \in \mathbb{R}^n$ and $t > 0$, then

\[
u(x, t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left( \frac{x - y}{t} \right) + g(y) \right\}.
\]

We call the expression on the right hand side of (10) the Hopf-Lax formula.

**Remark.** We can rewrite (11) as

\[
u(x, t) = \min_{y \in \mathbb{R}^n} \max_{z \in \mathbb{R}^n} \left\{ z \cdot (x - y) - tH(z) + g(y) \right\}
\]

Note that for each fixed $y, z$, the linear function

\[(x, t) \mapsto z \cdot (x - y) - tH(z) + g(y)\]

solves the PDE in (9). Thus Hopf’s formula builds a solution of (9) by taking appropriate “envelope” of such functions using minima and maxima.

**Proof.** (i) Fix any $y \in \mathbb{R}^n$ and define

\[
w(s) := y + \frac{8}{t}(x - y), \quad 0 \leq s \leq t.
\]

Then

\[
\int_0^t L(\dot{w}(s)) ds = tL\left( \frac{x - y}{t} \right).
\]

The definition (8) of $u$ implies

\[
u(x, t) \leq \int_0^t L(\dot{w}(s)) ds + g(y) = tL\left( \frac{x - y}{t} \right) + g(y),
\]

and so

\[
u(x, t) \leq \inf_{y \in \mathbb{R}^n} \left\{ tL\left( \frac{x - y}{t} \right) + g(y) \right\}.
\]
(ii) On the other hand, if $w(s)$ is any $C^1$ function satisfying $w(t) = x$, we have, by Jensen’s inequality
\[
L \left( \frac{1}{t} \int_0^t w(s) ds \right) \leq \frac{1}{t} \int_0^t L(\dot{w}(s)) ds.
\]
Thus if we write $y = x(0)$, we find
\[
tL \left( \frac{x - y}{t} \right) + g(y) \leq \int_0^t L(\dot{w}(s)) ds + g(y),
\]
and hence
\[
\inf_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\} \leq u(x, t).
\]

(iii) We have so far shown that
\[
(12) \quad u(x, t) = \inf_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\}.
\]
In order to show that the infimum is really a minimum, observe first that our setting $y = x$ in the expression on the right hand side of (12) gives the estimate
\[
(13) \quad u(x, t) \leq tL(0) + g(x).
\]
Now since
\[
\lim_{|q| \to \infty} \frac{L(q)}{|q|} = \infty
\]
there exists a constant $A$ such that
\[
L(q) \geq 2(Lip(g) + 1)|q| \quad \text{if } |q| \geq A.
\]
Consequently, if $|x - y| \geq tA$,
\[
tL \left( \frac{x - y}{t} \right) + g(y) \geq 2(Lip(g) + 1)|x - y| + g(y)
\]
\[
\geq (Lip(g) + 2)|x - y| + g(x)
\]
\[
\geq (Lip(g) + 2)|x - y| - tL(0) + u(x, t), \quad \text{by (13)}.
\]
Thus
\[
tL \left( \frac{x - y}{t} \right) + g(y) \geq u(x, t)
\]
if $|x - y| \geq tB$, for
\[
B \equiv \max \left[ A, \frac{L(0)}{Lip(g) + 1} \right].
\]
Hence (12) becomes
\[
(12) \quad u(x, t) = \min_{y \in B_{tB}(x)} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) \right\}. \quad \square
\]
• We now commence a study of the various properties of the function \( u \) defined by Hopf’s formula.

**Lemma 3.** The function \( u \) satisfies

\[
|u(x, t) - u(x, t)| \leq \text{Lip}(g)|\overline{x} - x|, \quad \forall t > 0, x, \overline{x} \in \mathbb{R}^n, 
\]

and

\[
u = g \quad \text{on} \; \mathbb{R}^n \times \{ t = 0 \}.
\]

**Proof.** (i) Fix \( t > 0, x, \overline{x} \in \mathbb{R}^n \). Choose \( y \in \mathbb{R}^n \) such that

\[
tL \left( \frac{x - y}{t} \right) + g(y) = u(x, t).
\]

Then

\[
u(x, t) - u(x, t) = \inf_z \left\{ tL \left( \frac{\overline{x} - z}{t} \right) + g(z) \right\} - tL \left( \frac{x - y}{t} \right) - g(y)
\]

\[
\leq g(\overline{x} - x + y) - g(y)
\]

\[
\leq \text{Lip}(g) |\overline{x} - x|.
\]

Interchanging the roles of \( \overline{x} \) and \( x \), we establish (14).

(ii) Now select \( x \in \mathbb{R}^n, t > 0 \). Setting \( y = x \) in the expression on the right hand side of (12) gives the estimate

\[
u(x, t) \leq tL(0) + g(x).
\]

Furthermore

\[
u(x, t) - g(x) = \min_{y \in \mathbb{R}^n} \left\{ tL \left( \frac{x - y}{t} \right) + g(y) - g(x) \right\}
\]

\[
\geq \min_{y \in \mathbb{R}^n} \left\{ -\text{Lip}(g)|x - y| + tL \left( \frac{x - y}{t} \right) \right\}
\]

\[
= -t \max_{z \in \mathbb{R}^n} \{ \text{Lip}(g)|z| - L(z) \}, \quad \text{setting} \; z = \frac{x - y}{t}
\]

\[
= -t \max_{w \in B_{\text{Lip}(g)}(0)} \max_{z \in \mathbb{R}^n} \{ w \cdot z - L(z) \}
\]

\[
= -t \max_{w \in B_{\text{Lip}(g)}(0)} H(w).
\]

This inequality and (15) imply \( |u(x, t) - g(x)| \leq Ct \) for

\[
C \equiv \max(L(0), \max_{B_{\text{Lip}(g)}(0)} H). \quad \Box
\]
Lemma 4. For each $x \in \mathbb{R}^n$ and $0 < s < t$,

\[ u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}. \tag{17} \]

Proof. \textbf{(i) Claim:} $\forall y \in \mathbb{R}^n$, $u(x,t) \leq (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$. Indeed, fix $t > 0$, $x \in \mathbb{R}^n$. Choose arbitrarily $y \in \mathbb{R}^n$. Then for some $z \in \mathbb{R}^n$,

\[ u(y,s) = sL\left(\frac{y-z}{s}\right) + g(z). \]

\[ \Rightarrow u(x,t) - u(y,s) \leq \left[ tL\left(\frac{x-z}{t}\right) + g(z) \right] - \left[ sL\left(\frac{y-z}{s}\right) + g(z) \right] \]

\[ = tL\left(\frac{x-z}{t}\right) - sL\left(\frac{y-z}{s}\right). \tag{18} \]

Now since $L$ is convex and

\[ \frac{x-z}{t} = \left(1 - \frac{s}{t}\right)\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s}, \]

we have

\[ L\left(\frac{x-z}{t}\right) \leq \left(1 - \frac{s}{t}\right)L\left(\frac{x-y}{t-s}\right) + \frac{s}{t}L\left(\frac{y-z}{s}\right). \]

Substituting this into (18), we obtain

\[ u(x,t) - u(y,s) \leq (t-s)L\left(\frac{x-y}{t-s}\right). \]

Therefore

\[ u(x,t) \leq \min_{y \in \mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}. \]

\textbf{(ii) Claim:} $\exists y \in \mathbb{R}^n$, $u(x,t) \geq (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$. For this, select $w \in \mathbb{R}^n$ such that

\[ u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w). \]

\[ \Rightarrow u(x,t) - u(y,s) \geq \left[ tL\left(\frac{x-w}{t}\right) + g(z) \right] - \left[ sL\left(\frac{y-w}{s}\right) + g(w) \right] \]

\[ = tL\left(\frac{x-w}{t}\right) - sL\left(\frac{y-w}{s}\right), \quad \forall y \text{ and } \forall s < t. \tag{19} \]

On the other hand, if we set $y = \frac{s}{t}x + (1 - \frac{s}{t})w$, then $\frac{x-y}{t-s} = \frac{x-w}{t} = \frac{x-w}{s}$, and hence

\[ tL\left(\frac{x-w}{s}\right) - sL\left(\frac{y-w}{s}\right) = (t-s)L\left(\frac{x-y}{t-s}\right). \]

Substituting this into (19), we obtain

\[ u(x,t) \geq tL\left(\frac{x-w}{t}\right) - sL\left(\frac{y-w}{s}\right) + u(y,s) \]

\[ \geq \min_{y \in \mathbb{R}^n} \left\{ (t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s) \right\}. \quad \square \]
Lemma 5. The function $u$ is Lipschitz on $\mathbb{R}^n \times [0, \infty)$.

Proof. By (14),

$$\text{Lip}(u(\cdot, \bar{t})) \leq \text{Lip}(g).$$

Hence it only remains to claim:

$$\forall x \in \mathbb{R}^n, 0 < \bar{t} < t, |u(x, t) - u(x, \bar{t})| \leq C|t - \bar{t}|,$$

for some constant $C$.

For this, select $x \in \mathbb{R}^n, 0 < \bar{t} < t$. Setting $y = x$ in the expression on the right hand side of (17) gives the estimate

$$u(x, t) \leq tL(0) + u(x, \bar{t}).$$

On the other hand, by (14) Consequently, (17), (20) and calculations like those employed in the proof of Lemma 3 gives

$$u(x, t) - u(x, \bar{t}) = \min_{y \in \mathbb{R}^n} \left\{ tL(\frac{x - y}{t}) + u(y, \bar{t}) - u(x, \bar{t}) \right\}$$

$$\geq \min_{y \in \mathbb{R}^n} \left\{ -\text{Lip}(g)|x - y| + tL(\frac{x - y}{t}) \right\}$$

$$= -t \max_{w \in B_{\text{Lip}(g)}(0)} H(w).$$

Hence (21) holds for the constant $C$ defined by (16). □

• Now Rademacher’s Theorem asserts that a Lipschitz function is differentiable almost everywhere.
  Consequently in view of Lemma 4 our function $u$ defined by Hopf’s formula (11) is differentiable a.e.

• The next theorem asserts that $u$ in fact solves the Hamilton-Jacobi equation a.e.

Theorem 5. Suppose $x \in \mathbb{R}^n, t > 0$, and $u$ is differentiable at $(x, t)$. Then

$$u_t(x, t) + H(Du(x, t)) = 0.$$

Proof. (i) Claim: $\forall q \in \mathbb{R}^n, u_t(x, t) + q \cdot Du(x, t) - L(q) \leq 0$.

Indeed, fix $q \in \mathbb{R}^n, h > 0$. Then by Lemma 4,

$$u(x + hq, t + h) = \inf_{y \in \mathbb{R}^n} \left\{ hL(\frac{x + hq - y}{h}) + u(y, t) \right\}$$

$$\leq hL(q) + u(x, t).$$

Hence

$$\frac{u(x + hq, t + h) - u(x, t)}{h} \leq L(q).$$
Letting \( h \to 0^+ \), we obtain

\[
  u_t(x, t) + q \cdot Du(x, t) \leq L(q).
\]

This inequality is valid for all \( q \in \mathbb{R}^n \) and so

\[
  u(x, t) + H(Du(x, t)) = u_t(x, t) + \sup_{q \in \mathbb{R}^n} \{ q \cdot Du(x, t) - L(q) \} \leq 0.
\]

(ii) **Claim:** \( \exists q \in \mathbb{R}^n \text{ s.t. } u_t(x, t) + q \cdot Du(x, t) - L(q) \geq 0. \)

For this, select \( z \in \mathbb{R}^n \) such that

\[
  u(x, t) = t L\left( \frac{x - z}{t} \right) + g(z),
\]

and for \( h > 0 \), set

\[
  s = t - h, \quad \text{and} \quad y = \frac{s}{t} x + \left( 1 - \frac{s}{t} \right) z.
\]

Then

\[
  \frac{x - y}{t - s} = \frac{x - z}{t} = \frac{y - z}{s}.
\]

Consequently,

\[
  u(x, t) - u(y, s) \geq t L\left( \frac{x - z}{t} \right) + g(z) - \left[ s L\left( \frac{y - z}{s} \right) + g(z) \right]
\]

\[
= \left( t - s \right) L\left( \frac{x - z}{t} \right);
\]

that is,

\[
  \frac{u(x, t) - u((x - h \frac{x - z}{t}), t - h)}{h} \geq L\left( \frac{x - z}{t} \right).
\]

Letting \( h \to 0^+ \), we obtain

\[
  u_t(x, t) + \frac{x - z}{t} : Du(x, t) \geq L\left( \frac{x - z}{t} \right).
\]

Consequently

\[
  u_t(x, t) + H(Du(x, t)) = u_t(x, t) + \max_{q \in \mathbb{R}^n} \{ q \cdot Du(x, t) - L(q) \}
\]

\[
\geq u_t(x, t) + \frac{x - z}{t} : Du(x, t) - L\left( \frac{x - z}{t} \right) \geq 0. \quad \square
\]

**Corollary 6.** The function \( u \) defined by Hopf’s formula (11) is differentiable a.e. in \( \mathbb{R}^n \times (0, \infty) \), and solves the initial value problem for Hamilton-Jacobi PDE

\[
  \begin{cases}
    u_t + H(u_x) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\
    u = g & \text{on } \mathbb{R}^n \times \{t = 0\}.
  \end{cases}
\]