The One-Dimensional Wave Equation

- Equation

\[(1) \quad u_{tt} - c^2(x, t) u_{xx} = f(x, t)\]

is called the one-dimensional wave equation.

- The coefficient \(c\) has the dimension of a speed and in fact, we will shortly see that it represents the wave propagation along the string.

- When \(f \equiv 0\), the equation is homogeneous and the superposition principle holds: if \(u_1\) and \(u_2\) are solutions of

\[(2) \quad u_{tt} - c^2 u_{xx} = 0\]

and \(a, b \in \mathbb{R}\), then \(au_1 + bu_2\) is also a solution.

- More generally, if \(u_k(x, t)\) is a family of solutions depending on the parameter \(k\) and \(g = g(k)\) is a function rapidly vanishing at infinity, then

\[\sum_{k=1}^{\infty} u_k(x, t) g(k) \quad \text{and} \quad \int_{-\infty}^{+\infty} u_k(x, t) g(k) dk\]

are still solutions of (2).

- Suppose we are considering the space-time region

\[0 < x < L, \quad 0 < t < T.\]

By analogy with the Cauchy problem for second order o.d.e., the second order derivative in (1) suggests that in a well-posed problem for the (one-dimensional) wave equation not only the initial profiles of the string but the initial velocity has to be assigned as well.

Thus, our initial (or Cauchy) data are

\[u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \quad x \in [0, L].\]

- The boundary data are typically:

(i) **Dirichlet data** describes the displacement of the end points of the string:

\[u(0, t) = a(t), \quad u(L, t) = b(t), \quad t > 0.\]

If \(a(t) = b(t) \equiv 0\) (homogeneous data), both ends are fixed, with zero displacement.

(ii) **Neumann data** describes the applied (scalar) vertical tension at end points. AS in the derivation of the wave equation, we may model this tension by \(\tau_0 u_x\), so that the Neumann conditions takes the form

\[\tau_0 u_x(0, t) = a(t), \quad \tau_0 u_x(L, t) = b(t), \quad t > 0.\]

In the special case of homogeneous data, \(a(t) = b(t) \equiv 0\), both ends of the string are attached to a frictionless sleeve and are free to move vertically.

(iii) **Robin data** describes a linear elastic attachment at end points.

One way to realize this type of boundary condition is to attach an end point to a linear spring whose other end is fixed.

- This translates into assigning

\[\tau_0 u_x(0, t) = ku(0, t), \quad \tau_0 u_x(L, t) = -ku(L, t), \quad t > 0,\]

where \(k > 0\) is the elastic constant of the spring.
1. The Global Cauchy problem for An Infinite String.

We may think of a string of infinite length and assign only initial data

\[ u(x, 0) = g(x), \quad u_t(x, 0) = h(x), \quad x \in \mathbb{R}. \]

Although physically unrealistic, it turns out that the solution of the global Cauchy problem is of fundamental importance.

The solution of the global Cauchy problem is given by the d’Alembert formula

\[ u(x, t) = \frac{1}{2} [g(x - ct) + g(x + ct)] + \int_{x-ct}^{x+ct} h(y) dy. \]

2. The Semifinite String.

The initial-boundary problem is

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u(x, t) - c^2 \frac{\partial^2}{\partial x^2} u(x, t) &= 0, \quad x > 0, \quad t > 0 \\
u(x, 0) &= g(x), \quad u_t(x, 0) = h(x), \quad x \geq 0 \\
u_0(t) &= 0 \quad (\text{fixed endpoint}), \quad t \geq 0 \\
u_x(0, t) &= 0 \quad (\text{free endpoint}), \quad t \geq 0,
\end{align*}
\]

where \(c^2 = \frac{\tau_0}{\rho_0}\) is constant.

- The problem for the semi-infinite string can be reduced to the problem for the infinite string, so that the solution of the problem for the infinite string, when restricted to a half-line, yields the solution of the problem for the semi-infinite string.

- To do this, we must extend the initial condition to the entire line in such a way that the solution satisfies the boundary conditions at the point \(x = 0\).

- Parity considerations are helpful in doing this.

Case I. The initial condition can be extended to the entire line as an even function provided the relation

\[ u_x \bigg|_{x=0} = 0 \]

holds.

But will the solution be an even function of \(x\) at all times?

- The wave equation is invariant under the transformation

\[ x \mapsto -x. \]

If the initial condition is even, that is, also invariant under this transformation, we then have two solutions \(u(x, t)\) and \(u(-x, t)\) of a global Cauchy problem.

- However, the uniqueness of the solution of the global Cauchy problem was proved in the derivation of the d’Alembert formula.
Hence the two solutions $u(x, t)$ and $u(-x, t)$ coincide, so that

$$u(x, t) = u(-x, t),$$

and the solution is an even function.

- This is a general idea: If the problem possesses some symmetry and the solution is unique, then the solution must also possess the symmetry.

Case II. We can use an odd extension when the condition $u\big|_{x=0} = 0$ holds.

The Cauchy-Dirichlet Problem for The Finite String.

Suppose that the vibration of a violin chord is modelled by the following Cauchy-Dirichlet problem

$$\begin{cases}
    u_{tt} - c^2 u_{xx} = 0, & 0 < x < L, \ t > 0 \\
    u(x, 0) = g(x), \ u_t(x, 0) = h(x), & 0 \leq x \leq L, \\
    u(0, t) = u(L, t) = 0, & t \geq 0,
\end{cases}$$

where $c^2 = \tau_0 / \rho_0$ is constant.

- The d’Alembert’s method is not very convenient for solving the boundary value problem in the case of finite string. Later on we will develop another very powerful method to handle this case.

- At present we illustrate the applications of d’Alembert’s method to the problem

$$\begin{cases}
    u_{tt} - c^2 u_{xx} = 0, & 0 < x < L, \ t > 0 \\
    u(x, 0) = g(x), \ u_t(x, 0) = h(x), & 0 \leq x \leq L, \\
    u(0, t) = k_1(t), \ u(L, t) = 0, & t \geq 0,
\end{cases}$$

The values brought from the boundary and the initial interval $t = 0$ along the characteristics $x \pm ct = \text{constant}$ contribute to the solution at the point $(x, t)$. The characteristics undergo bending as they reflected from the boundary.

- As a result, the value of the solution is an alternating sum of the values at the nodes of the resulting broken lines.

Uniqueness. Use conservation of energy.

- Let $u$ and $v$ be solutions of

$$\begin{cases}
    u_{tt} - c^2 u_{xx} = f(x, t), & 0 < x < L, \ t > 0 \\
    u(x, 0) = g(x), \ u_t(x, 0) = h(x), & 0 \leq x \leq L, \\
    u(0, t) = k_1(t), \ u(L, t) = k_2(t), & t \geq 0,
\end{cases}$$

Then

$$w = u - v$$
is a solution of the problem

\[
\begin{cases}
  u_{tt} - c^2 u_{xx} = 0, & 0 < x < L, \ t > 0 \\
  u(x,0) = 0, \ u_t(x,0) = 0, & 0 \leq x \leq L, \\
  u(0,t) = 0, \ u(L,t) = 0, & t \geq 0.
\end{cases}
\]

**Claim:** \( w \equiv 0. \)

- The total mechanical energy

\[
E(t) = E_{\text{kin}}(t) + E_{\text{pot}}(t) = \frac{1}{2} \int_0^L \left[ \rho_0 w_t^2 + \tau_0 w_x^2 \right] dx
\]

and in our case we have

\[
\dot{E}(t) = 0
\]

since \( f = 0 \) and \( w_t(x,0) = w_x(x,0) = 0 \), whence

\[
E(t) = E(0), \ \forall t \geq 0.
\]

Since, on particular, \( w_t(x,0) = w_x(x,0) = 0 \), we have

\[
E(t) = E(0) = 0, \ \forall t \geq 0.
\]

On the other hand, \( E_{\text{kin}}(t) \geq 0 \), \( E_{\text{pot}}(t) \geq 0 \), so that we deduce

\[
E_{\text{kin}}(t) = 0, \quad E_{\text{pot}}(t) = 0,
\]

which forces \( w_t = w_x = 0 \). Therefore \( w \) is constant.

- Since \( w(x,0) = 0 \), we conclude that

\[
w(x,t) = 0, \ \forall t \geq 0.
\]

**Remark.**

(i) If we pluck a violin chord at a point, the initial profile is continuous but has a corner at that point and cannot be even \( C^1 \).

The physically realistic assumption for the initial profile \( g \) is **continuity**.

(ii) Similarly, if we model the vibration of a chord set into motion by a strike or a little hammer, we should allow **discontinuity** in the initial velocity.

- Observe that d’Alembert formula makes perfect sense even for \( g \) **continuous** and \( h \) **bounded**.

  - The question is in which sense the resulting function satisfies the wave equation, since, in principle, it is not even differentiable, only continuous.

- It is possible to introduce **weak** formulations of the various initial-boundary value problem, in order to include realistic initial data and solutions with a low degree of regularity.