Systems of First Order PDEs

• For an ODE

\[ u'(x) = f(x, u(x)), \]

we found that the existence of solutions was no harder to prove for a function
\( u : \mathbb{R} \to \mathbb{R}^n \) than it was for the case of a function \( u : \mathbb{R} \to \mathbb{R} \).

Namely, we could consider (1) to be a system of equations

\[ u'_i(x) = f_i(x, u_1(x), \ldots, u_n(x)). \]

This enables us to solve an \( n \)-th order equation

\[ u^n(x) = f(x, u(x), u'(x), \ldots, u^{(n-1)}(x)), \]

since equation (2) is equivalent to the system of equations

\[
\begin{cases}
  u' = u_1 \\
  u'_1 = u_2 \\
  \vdots \\
  u'_{n-2} = u_{n-1} \\
  u'_{n-1} = f(x, u(x), u'(x), \ldots, u^{(n-1)}(x)).
\end{cases}
\]

More precisely, if \( u \) satisfies (2), then \((u, u', \ldots, u^{(n-1)})\) satisfies (3); conversely, if \((u, u', \ldots, u^{(n-1)})\) satisfies (3), then \( u \) satisfies (2) and moreover \( u_i = u^{(i)} \).

Since (3) can be solved with any initial conditions \((u(x_0), \ldots, u_{n-1}(x_0))\), equation (2) can be solved with any initial conditions \(u(x_0), u'(x_0), \ldots, u^{(n-1)}(x_0)\).

• There is no such general theorem about system of first order PDEs.

Because of the considerations in the previous section, we will assume that the partials of \( u \) with respect to one of the variables, which we call \( x_1, \ldots, x_n \). Thus we consider the equation

\[ \frac{\partial^k u}{\partial x_1^j \cdots \partial x_n^j}(x) = f\left(x, u(x), \frac{\partial^l u}{\partial x_1^l \cdots \partial x_n^l}, \ldots\right); \]

the partial derivatives appearing on the right are all of order \( \ell \leq k \), and the order \( j \) with respect to \( y \) is \( \leq k - 1 \).

Second Order Equations in 2 Variables

• In order to make the situation less unwieldy, let us take the simple, but completely representative, case of a second order equation in 2 variables

\[ \frac{\partial^2 u}{\partial y^2}(x, y) = f(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y)), \]

with the initial conditions

\[
\begin{cases}
  u(x, 0) = \xi(x) \\
  \frac{\partial u}{\partial y}(x, 0) = \eta(x).
\end{cases}
\]

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To reduce this to a system of first order equations, we introduce new unknown functions

\[ p_1, p_2, p_{11}, p_{12}, p_{22} \]

for \( u_x, u_y, u_{xx}, u_{xy}, u_{yy} \).

We will denote the various partial derivatives of \( f \) in (4) by

\[ f_x, f_y, f_u, f_{p1}, f_{p2}, f_{p11}, f_{p12} \]

Consider the following system of equations for \( p_1, p_2, p_{11}, p_{12}, p_{22} \)

\[
\begin{align*}
(i) \quad & \frac{\partial u}{\partial y} = p_2 \\
(ii) \quad & \frac{\partial p_1}{\partial y} = p_{12} \\
(iii) \quad & \frac{\partial p_2}{\partial y} = p_{22} \\
(iv) \quad & \frac{\partial p_{11}}{\partial y} = \frac{\partial p_{12}}{\partial x} \\
(v) \quad & \frac{\partial p_{22}}{\partial y} = f_y + f_u \cdot p_2(x, y) + f_{u_x} \cdot p_{12}(x, y) + f_{u_y} \cdot p_{22}(x, y) \\
& + f_{u_{xx}} \cdot \frac{\partial p_{12}}{\partial x}(x, y) + f_{u_{xy}} \cdot \frac{\partial p_{22}(x, y)}{\partial x},
\end{align*}
\]

where the last equation is obtained by differentiating (4) with respect to \( y \), and all partials of \( f \) are evaluated at \((x, y, u(x, y), p_1(x, y), p_2(x, y), p_{11}(x, y), p_{12}(x, y))\).

This system expresses partials with respect to \( y \) in terms of partials with respect to \( x \) alone;
it was precisely in order to achieve this that we had to introduce new unknowns for all the partials of \( u \) up to order 2, unlike the case of a second order ODE, where we only introduce the first derivatives as a new unknown.

This system is first order and also \textbf{quasilinear}.

If \( u \) is a \( C^3 \) function satisfying (4), then \((u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\) satisfies (6).

However, it is \textbf{not} true that every solution \((u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})\) of (6) has \( u \) satisfying (4).

- For one thing, the first five equations do not even allow us to identify \( p_1, p_2, p_{11}, p_{12}, p_{22} \) with \( u_x, u_y, u_{xx}, u_{xy}, u_{yy} \).
- Equations (i) and (iii) give
  \[ p_2 = u_y, \text{ and thus } p_{22} = \frac{\partial p_2}{\partial y} = u_{yy}, \]
  but equation (v), for example, gives only
  \[ \frac{\partial p_{12}}{\partial y} = \frac{\partial p_{22}}{\partial x} = u_{yy} = u_{xyx}, \]
  and hence
  \[ p_{12}(x, y) = u_{xy}(x, y) + A(x), \]
  for an arbitrary function \( A \).
- Moreover, even if we know that the \( p \)'s were the partial derivatives of \( u \), equation (vi) would still not imply equation (i), for the two sides could differ by an arbitrary function of \( x \).
On the other hand, there is a Cauchy problem for the system (6) which is equivalent to the Cauchy problem (4)(5); namely, if \( u, p_1, p_2, p_{11}, p_{12}, p_{22} \), satisfies (6) with the initial conditions

\[
\begin{aligned}
&u(x, 0) = \xi(x) \\
&p_1 = \xi'(x) \\
&p_2 = \eta(x) \\
&p_{11} = \xi''(x) \\
&p_{12} = \eta'(x) \\
&p_{22}(x, 0) = f(x, 0, \xi(x), \xi'(x), \eta(x), \xi''(x), \eta'(x)),
\end{aligned}
\]

and \( u \) is \( C^3 \), then \( u \) satisfies (4) with the initial condition (5) and moreover \( u_x = p_1, u_y = p_2, u_{xx} = p_{11}, u_{xy} = p_{12}, u_{yy} = p_{22} \).

\( \circ \) To prove this, we note first that equations (i) and (iii) of (6) give

\[
\begin{aligned}
p_{22} &= u_y, \\
p_{22} &= u_{yy},
\end{aligned}
\]

- The equation (v) of (6) gives

\[
\frac{\partial p_{12}}{\partial y} = \frac{\partial u_{yy}}{\partial x} = \frac{\partial u_{xy}}{\partial y},
\]

while the initial conditions give

\[
p_{12}(x, 0) = \eta'(x) = \frac{\partial p_2}{\partial x}(x, 0) = \frac{\partial u_y}{\partial x}(x, 0) = u_{xy}(x, 0),
\]

so that we must have

\[
p_{12} = u_{xy}.
\]

- The equation (ii) of (6) gives

\[
\frac{\partial p_1}{\partial y} = p_{12} = u_{xy}, \quad \text{by (9)}
\]

while the initial conditions give

\[
p_1(x, 0) = \xi'(x) = u_x(x, 0),
\]

so that we must have

\[
p_1 = u_x.
\]

- Finally, the equation (iv) of (6) and (9) give

\[
\frac{\partial p_{11}}{\partial y} = \frac{\partial p_{12}}{\partial x} = \frac{\partial u_{xy}}{\partial x} = \frac{\partial u_{xx}}{\partial y}.
\]
while the initial conditions give
\[ p_{11}(x,0) = \xi''(x) = \frac{\partial p_1}{\partial x}(x,0) = u_{xx}(x,0), \quad \text{by (10)}, \]
so that we must have
\[ p_{11} = u_{xx}. \]

Equation (vi) of (6) now shows that the two sides of (4) have the same partial
derivatives with respect to \( y \).
The initial conditions then imply that the two sides are equal.

**Higher Order Equations in \( n + 1 \) Variables**

- Exactly the same procedure, but with considerably more complicated notation,
proves that the equation
\[
\frac{\partial^k u}{\partial y^k}(x_1, \cdots, x_n, y) = f(x_1, \cdots, x_n, y, u(x_1, \cdots, x_n, y), \cdots, \frac{\partial^\ell u}{\partial x^{i_1}\cdots\partial x^{i_n}\partial y^j}, \cdots)
\]
with the initial conditions
\[
\begin{cases}
  u(x_1, \cdots, x_n, 0) = \xi_0(x_1, \cdots, x_n) \\
  \frac{\partial^{k-1} u}{\partial x^m_n}(x_1, \cdots, x_n, 0) = \xi_{k-1}(x_1, \cdots, x_n)
\end{cases}
\]
is equivalent to a system of first order quasi-linear equations with initial conditions.

- The equations expresses the partial derivatives of the unknown functions with
respect to \( y \) in terms of partial derivatives with respect to \( x_1, \cdots, x_n \);
the number of unknown functions will be the number of distinct derivatives
\[
\frac{\partial^\ell u}{\partial x^{i_1}\cdots\partial x^{i_n}\partial y^j}, \quad \text{with } 0 \leq \ell \leq k, \text{ and } j \leq k - 1.
\]

**Systems of Equations in \( n + 1 \) Variables**

- Similar procedures allow us to reduce a Cauchy problem for a (not necessarily
quasi-linear) system of equations to the Cauchy problem for a quasi-linear system
in more unknowns.
  - To take a simple example, consider a first order system
\[
\begin{align*}
  u_i^j &= F_i^j(x, y, \cdots, u_i, \cdots, u_k^j, \cdots), \\
  u_i^j(x, 0) &= \xi_i^j(x),
\end{align*}
\]
with initial conditions
we simply construct the new system
\[
\begin{cases}
  u_i^j = F_i^j(x, y, \cdots, u_i, \cdots, p_i^j, \cdots) \\
  p_i^j = F_i^j + \sum_j F_{u_i}^j \cdot u_k^j + \sum_j F_{p_j}^i \cdot p_k^j
\end{cases}
\]
with initial conditions
\[
\begin{cases}
  u_i^j(x, 0) = \xi_i^j(x), \\
  p_i^j(x, 0) = \xi_i^j(x).
\end{cases}
\]
This system is quasi-linear, and the solution \( \{u^i, p^i\} \) clearly gives us a solution
\( \{u^i\} \) of the original system.