Compactness, Fredholm Alternative and Spectrum

I. Compactness

Definition. We say that \( E \subseteq X \) is compact if from every open covering of \( E \) it is possible to extract a finite subcovering of \( E \).

Definition. A subset \( E \) of a normed space \( X \) is sequentially compact if for every sequence \( \{x_k\} \subset E \) there exists a subsequence \( \{x_k\} \), converging in \( X \).

Theorem 1. Let \( X \) be a normed space and \( E \subset X \). Then \( E \) is compact iff it is sequentially compact.

- While a compact set is always closed and bounded, the following example exhibits a closed and bounded set which is not compact in \( \ell^2 \).

Example. Consider the real Hilbert space

\[
\ell^2 = \{ x = \{ x_k \}_{k=1}^\infty : \sum_{k=1}^\infty x_k^2 < \infty, \ x_k \in \mathbb{R} \}
\]

equipped with

\[
\langle x, y \rangle = \sum_{k=1}^\infty x_k y_k, \quad \text{and} \quad \| x \|^2 = \sum_{k=1}^\infty x_k^2.
\]

Let \( E = \{ e_k \}_{k \geq 1} \), where \( e^1 = \{1, 0, 0, \cdots\} \), \( e^2 = \{0, 1, 0, \cdots\} \), etc.

Observe that \( E \) constitute an orthonormal basis in \( \ell^2 \). Thus, \( E \) is closed and bounded in \( \ell^2 \). However, \( E \) is not sequentially compact. Indeed, \( \| e_i - e_j \| = \sqrt{2} \), if \( j \neq k \), and therefore no subsequence of \( \{e_k\}_{k \geq 1} \) can be convergent.

- In fact, only in finite-dimensional spaces each closed and bounded set is compact.

Theorem 1. Let \( B \) be a Banach space. \( B \) is finite-dimensional iff the unit ball \( \{ x : \| x \| \leq 1 \} \) is compact.

II. Weak Convergence and Compactness.

Definition. Let \( H \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). A sequence \( \{x_k\} \subset H \) converges weakly to \( x \in H \) and we write \( x_k \rightharpoonup x \), if

\[
\langle F, x_k \rangle \to \langle F, x \rangle, \quad \forall F \in H^*.
\]

- The convergence in norm is called strong convergence.

- From Riesz’s representation theorem, it follows that \( \{x_k\} \subset H \) converges weakly to \( H \) iff

\[
\langle x_k, y \rangle \to \langle x, y \rangle, \quad \forall y \in H.
\]

- The weak limit is unique, since \( x_k \to x \) and \( x_k \to z \) imply

\[
\langle x - z, y \rangle, \quad \forall y \in H,
\]

whence \( x = z \).
• **Strong convergence implies weak convergence**, since Schwarz’s inequality gives
  \[ \| \langle x_k, y \rangle \| \leq \| x_k - x \| \| y \|. \]
  - The two notions of convergence are equivalent in the finite-dimensiona spaces.
  - It is not so in infinite dimensions, as the following example shows.

**Example.** Let \( H = L^2(0, 2\pi) \). The sequence \( v_k = \cos kx, \ k \geq 1 \), is **weakly convergent to zero**. In fact, \( \forall f \in L^2(0, 2\pi) \), the Riemann-Lebesgue theorem on the Fourier coefficients of \( f \) implies that
  \[ \langle f, v_k \rangle = \int_{2\pi}^0 f(x) \cos kx \, dx \to 0, \ \text{as} \ k \to \infty. \]

However
  \[ \| u_k \|_{L^2(0, 2\pi)} = \sqrt{\pi} \]
and therefore \( \{ v_k \}_{k \geq 1} \) does not converge strongly.

**Example.** Let \( \{ \varphi \}_{i=1,2,\ldots} \) denote an orthonormal system in the Hilbert space \( \mathcal{H} \), and we observe
  \[ \| \varphi_i - \varphi_j \| = \sqrt{2}, \ \forall i, j \in \mathbb{N}, \ i \neq j. \]
Consequently \( \{ \varphi_i \}_{i=1,2,\ldots} \) does not contains a strongly convergent subsequence.

According to Bessel’s inequality,
  \[ \sum_{i=1}^{\infty} |\langle \varphi, f \rangle|^2 \leq \| f \|^2 < \infty, \ \forall f \in \mathcal{H}, \]
and we infer
  \[ \lim_{i \to \infty} \langle \varphi_i, f \rangle = 0 = \langle 0, f \rangle \ \forall f \in \mathcal{H}. \]
We therefore obtain \( \varphi \to 0 \) as \( i \to \infty \). Note that
  \[ \| 0 \| < 1 = \lim_{k \to \infty} \| \varphi \|. \]

**Principle of Uniform Boundedness.** On the Hilbert space \( H \) let the sequence of bounded functionals \( A_n : H \to \mathbb{C} \) with \( n \in \mathbb{N} \) be given, such that each element \( f \in H \) possesses a constant \( c_f \in [0, +\infty) \) with the property
  \[ |A_n f| \leq c_f, \ \ n = 1, 2, \ldots. \]
Then we have a constant \( \alpha \in [0, \infty) \) satisfying
  \[ \| A_n \| \leq \alpha, \ \forall n \in \mathbb{N}. \]

**Proof. Step 1.** Choose \( f_0 \in H, \ \varepsilon > 0 \) and a constant \( c \geq 0 \). Let \( A : H \to \mathbb{C} \) denote a bounded linear functional such that
  \[ |Af| \leq c, \ \forall f \in H \ \text{with} \ \| f - f_0 \| \leq \varepsilon. \]
Then we have
  \[ \| A \| \leq \frac{2c}{\varepsilon}; \]
indeed, if \( \| x \| \leq 1, \ \exists f \) with \( \| f - f_0 \| \leq \varepsilon \) such that \( x = \frac{1}{\varepsilon}(f - f_0) \), and we infer
  \[ |Ax| = \left| \frac{1}{\varepsilon}Af - \frac{1}{\varepsilon}Af_0 \right| \leq \frac{1}{\varepsilon}(|Af| + |Af_0|) \leq \frac{2c}{\varepsilon}. \]
Step 2. If the statement (2) does not hold true, Step 1 together with the continuity of the functional \( \{ A_n \} \) enables us to construct a sequence of balls

\[
\Sigma_n := \{ f \in H : \| f - f_n \| \leq \varepsilon_n \}, \quad n \in \mathbb{N}
\]
satisfying \( \Sigma_1 \supset \Sigma_2 \supset \cdots \) with \( \varepsilon_n \searrow 0 \) as \( n \to \infty \) and an index-sequence \( 1 < n_1 < n_2 < \cdots \) such that

\[
|A_n x| \geq j, \quad \forall x \in \Sigma_j, \quad j \in \mathbb{N},
\]
which yields a contradiction to (1). \( \square \)

- We have observed that the norm in a Hilbert space is strongly continuous.
- With respect to weak convergence, the norm is only lower semincontinuous, as property (b) in the following theorem shows.

**Theorem 2.** Let \( \{ x_k \} \subset H \) such that \( x_k \) weakly converges to \( x \). Then

(a) \( \{ x_k \} \) is bounded,
(b) \( \| x \| \leq \liminf_{k \to \infty} \| x_k \| \).

*Proof.* (a) follows from the principle of uniform boundedness. For (b), it is enough to observe that

\[
\| x \|^2 \leq \lim_{k \to \infty} \langle x_k, x \rangle \leq \| x \| \liminf_{k \to \infty} \| x_k \|. \quad \square
\]

- The usefulness of weak convergence is revealed by the following compactness result.

**Theorem 3.** Every bounded sequence in a Hilbert space \( H \) contains a subsequence which is weakly convergent to an element \( x \in H \).

*Proof.* **Step 1.** The sequence \( \{ x_j \} \) is bounded and thus we have a constant \( c \in [0, +\infty) \) such that \( \| x_j \| \leq c, \quad j = 1, 2, \cdots \). On account of

\[
|\langle x_1, x_j \rangle| \leq c \| x_1 \|, \quad \forall j \in \mathbb{N},
\]

we find a subsequence \( \{ x_j^{(1)} \} \subset \{ x_j \} \) such that \( \lim_{j \to \infty} \langle x_1, x_j^{(1)} \rangle \) exists.
- Noting that

\[
|\langle x_2, x_j^{(1)} \rangle| \leq c \| x_2 \|, \quad \forall j \in \mathbb{N},
\]

we select a further subsequence \( \{ x_j^{(2)} \} \subset \{ x_j^{(1)} \} \) such that \( \lim_{j \to \infty} \langle x_2, x_j^{(2)} \rangle \) exists.
- The continuation of this process gives a chain of subsequence

\[
\{ x_j \} \supset \{ x_j^{(1)} \} \supset \{ x_j^{(2)} \} \supset \cdots \supset \{ x_j^{(k)} \}
\]

such that

\[
\lim_{j \to \infty} \langle x_i, x_j^{(k)} \rangle \quad \text{exists for } i = 1, \cdots, k.
\]

- The Cantor’s diagonal procedure then gives us the sequence \( x_j^{(k)} \) such that

\[
\lim_{k \to \infty} \langle x_i, x_j^{(k)} \rangle \quad \text{exists for all } i \in \mathbb{N}.
\]
Step 2. Denote the linear subspace of all finite linear combinations of $x_j$ by $\mathcal{M}$, namely

$$\mathcal{M} = \{ x : x = \sum_{i=1}^{N(x)} \alpha_i x_i, \quad \alpha_i \in \mathbb{C}, \quad N(x) \in \mathbb{N} \}.$$ 

Then

$$\lim_{k \to \infty} \langle x, x^{(k)} \rangle$$

exists for all $x \in \mathcal{M}$.

Making the transition to the linear closed subspace $\mathcal{M} \subset \overline{\mathcal{M}} \subset H$,

$$\lim_{k \to \infty} \langle y, x^{(k)} \rangle$$

exists for all $y \in \overline{\mathcal{M}}$.

Step 3. Due the projection theorem, each element $y \in H$ can be represented in the form

$$y = y_1 + y_2, \quad \text{with } y_1 \in \overline{\mathcal{M}}, \quad y_1 \in \mathcal{M}^\perp.$$ 

This implies the existence of

$$\lim_{k \to \infty} \langle y, x^{(k)} \rangle = \lim_{k \to \infty} \langle y_1 + y_2, x^{(k)} \rangle = \lim_{k \to \infty} \langle y_1, x^{(k)} \rangle, \quad \forall y \in H.$$ 

Step 4. From Step 3, we may define a linear functional $A$ in $H$ by setting

$$Ay = \lim_{k \to \infty} \langle y, x^{(k)} \rangle, \quad \forall y \in H.$$ 

Consider the bounded linear functional

$$A_i(y) = \langle x^{(i)}_i, y \rangle, \quad y \in H.$$ 

The principle of uniform boundedness gives us a constant $C \in [0, \infty)$ such that

$$\|x^{(n)}_n\| = \|A_n\| \leq C, \quad \forall n \in \mathbb{N}.$$ 

This and Theorem 1(b) imply $\|A\| \leq C$. Thus $A$ is a bounded linear functional and Riesz representation theorem then gives us an element $x \in H$ satisfying

$$A(y) = \langle x, y \rangle, \quad \forall y \in H.$$ 

We then obtain

$$\lim_{k \to \infty} \langle y, x^{(k)} \rangle = \lim_{k \to \infty} A_k(y) = A(y) = \langle x, y \rangle, \quad \forall y \in H,$$

which means that the sequence $\{x^{(k)}_k\}$ converges weakly. □

III. Compact Operators

- Let $V_1$ and $V_2$ be normed linear spaces. Every operator in $\mathcal{L}(V_1, V_2)$ transforms bounded sets in $V_1$ into bounded sets in $V_2$.

  The subclass of operators that transform **bounded** sets into **pre-compact** sets is particularly important.
Definition. A subset $\Sigma$ of a normed linear space $V$ is called precompact, if each sequence $\{y_m\} \subset \Sigma$ contains a strongly convergent subsequence $\{y_{m_k}\}$, which means
$$\lim_{k,\ell \to \infty} \|y_{m_k} - y_{n_\ell}\| = 0.$$  

Definition. Let $V_1$ and $V_2$ be normed linear spaces and $L \in \mathcal{L}(V_1, V_2)$. $L$ is said to be compact if for every bounded subset $E$ of $V_1$, the image $L(E)$ is pre-compact in $V_2$. This means that each sequence $\{x_m\}$ in $V_1$ with $\|x_m\|_{V_1} \leq r$, for some $r \in \mathbb{R}$ and all $n \in \mathbb{N}$, contains a subsequence $\{x_{m_k}\}$ such that $\{Lx_{m_k}\} \subset V_2$ converges strongly.

Proposition 4. A compact operator $K : V_1 \to V_2$ is bounded.

Proof. If $K : V_1 \to V_2$ were not bounded, there would exist a sequence $\{x_n\} \subset V_1$ with $\|x_n\| = 1$, for all $n \in \mathbb{N}$ and $\|Kx_m\|_2 \to \infty$ as $n \to \infty$. Therefore, we cannot select a convergent subsequence from $\{Kx_n\}$ in $V_2$. □

- An equivalent characterization of compact operators between Hilbert spaces may be given in terms of weak convergence. Indeed, an operator is compact iff it converts weak convergence to strong convergence. Precisely, we have the following.

Proposition 5. Let $H_1$ and $H_2$ be Hilbert spaces and $L \in \mathcal{L}(H_1, H_2)$. $L$ is compact iff for every sequence $\{x_k\} \subset H_1$,
$$x_k \to x \text{ in } H_1 \text{ weakly implies } Lx_k \to Lx \text{ in } H_2 \text{ strongly.}$$

Proof. ($\Rightarrow$) Let $\{x_m\} \subset H_1$ be a sequence with $\|x_m\| \leq 1$, $m \in \mathbb{N}$. According to Theorem 3, we have a subsequence $\{x_{m_k}\} \subset \{x_m\}$ which converges weakly to $x$ in $H_1$ as $k \to \infty$. Then, by assumption
$$Kx_{m_k} \to Kx \text{ strongly, } k \to \infty.$$  
Consequently, the operator $K : H_1 \to H_2$ is compact.

($\Leftarrow$) Now let $K$ be compact and $\{x_m\}$ denote a sequence weakly converging to $x$. We then have to prove $Kx_m \to Kx$ strongly as $m \to \infty$ in $H_2$.

If the latter statement were false, there would exist a number $d > 0$ and a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ satisfying
$$\|Kx_{m_k} - Kx\| \geq d > 0, \forall n \in \mathbb{N}.$$  
- On the other hand, by Theorem 2(b) the sequence $\{x_m\}$ is bounded in $H_1$. Since the operator is compact, we have a further subsequence $\{\tilde{x}_{m_k}\} \subset \{x_{m_k}\}$ with $K\tilde{x}_{m_k} \to y$ strongly, as $n \to \infty$.
- Moreover, $\{x_m\}$ converges weakly to $Kx$, which implies $y = Kx$. Hence $\|Kx_{m_k} - Kx\| \to 0$ as $m_k \to \infty$, in contradiction with (3). □

Example. From Theorem 3, the identity operator $I : H \to H$ is compact iff $\dim H < \infty$.

Also, any bounded operator with finite dimensional range is compact.

- The following proposition is useful.
**Proposition 6.** Let $L: H_1 \rightarrow H_2$ be compact. If $G \in \mathcal{L}(H_2,H_3)$ or $G \in \mathcal{L}(H_0,H_1)$, then the operator $G \circ L$ or $L \circ G$ is compact.

**Proof.** Since $L$ is compact, the sequence $x_k \rightharpoonup x$ weakly in $H_1$ as $k \rightarrow \infty$ is transformed into the sequence $Lx_k$ which converges strongly to $Lx$ in $H_2$ as $k \rightarrow \infty$.

(i) If $G: H_2 \rightarrow H_3$ is continuous, we infer
$$G \circ Lx_k \rightarrow G \circ Lx \quad \text{as} \quad n \rightarrow \infty \quad \text{in} \quad H_3.$$

(ii) If $G: H_0 \rightarrow H_1$ is continuous, the sequence $y_k \rightharpoonup y$ weakly in $H_0$ as $k \rightarrow \infty$ is transformed into the sequence $Gy_k$ which converges weakly to $Gy$ in $H_1$ as $n \rightarrow \infty$. Since $L$ is compact, the sequence $L \circ Gy_k$ converges strongly to $L \circ Gy$ in $H_2$ as $k \rightarrow \infty$. □

**IV. The Fredholm Alternative**

The Fredholm alternative concerns compact linear operators from a space $V$ into itself and is an extension of the theory of linear mappings in finite dimensional spaces.

- The result we are going to present are extensions of well known facts concerning the solvability of linear algebraic systems of the form

$$Ax = b,$$

where $A$ is an $n \times n$ matrix and $b \in \mathbb{R}^n$.

- The following dichotomy holds: either (4) has a unique solution for every $b$ or the homogeneous equation $Ax = 0$ has nontrivial solutions.

- More precisely, system (4) is solvable iff $b$ belong to the column space of $A$, which is the orthogonal complement of $\ker(A^\top)$.

- If $w_1, w_s$ span $\ker(A^\top)$, this amounts to asking the $s$ compatibility conditions,

$$b \cdot w_j = 0, \quad j = 1, \ldots, s.$$

Finally, $\ker(A)$ and $\ker(A^\top)$ have the same dimension and if $v_1, \ldots, v_n$ span $\ker(A)$, the general solution of (4) is given by

$$x = \overline{x} + \sum_{j=1}^{s} c_j v_j$$

where $\overline{x}$ is a particular solution of (4) and $c_1, \ldots, c_s$ are arbitrary constants.

- The extension to infinite-dimensional spaces requires some care.

**Theorem 7.** Let $K$ be a compact linear mapping of a normed linear space $V$ into itself. Then either

(i) the homogeneous equation $x - Kx = 0$ has a nontrivial solution $x \in V$ or

(ii) for each $y \in V$ the equation $x - Kx = y$ has a unique solution $x \in V$.

Furthermore, in case (ii), the operator $(I - K)^{-1}$ whose existence is asserted there is also bounded.

To prove Theorem 7, we first establish the following.
Proposition 8. Let $K$ be a compact linear mapping of a normed linear space $V$ into itself. Let $\Phi = I - K$. Then $\exists$ a constant $C$ such that

$$\text{dist}(x, \mathcal{N}(\Phi)) \leq C\|\Phi x\|, \quad \forall v \in V.$$ 

Proof. Indeed, suppose that the result is not true. Then there exists a sequence $\{x_n\} \subset V$ satisfying $\|\Phi x_n\| = 1$ and $d_n = \text{dist}(x_n, \mathcal{N}(\Phi)) \to \infty$.

- Choose a sequence $\{y_n\} \subset \mathcal{N}(\Phi)$ such that $d_n \leq \|x_n - y_n\| \leq 2d_n$. Then if $z_n = \frac{x_n - y_n}{\|x_n - y_n\|}$

we have $\|z_n\| = 1$ and $\|\Phi z_n\| \leq d_n^{-1} \to 0$, so that the sequence $\{\Phi z_n\}$ converges to 0.

- But since $K$ is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{Kz_n\}$ converges to an element $y_0 \in V$.

- Since $z_n = (\Phi + K)z_n$, we then also have $\{z_n\}$ converging to $y_0$ and hence $y_0 \in \mathcal{N}(\Phi)$.

$$\text{dist}(z_n, \mathcal{N}(\Phi)) = \inf_{y \in \mathcal{N}(\Phi)} \|z_n - y\|
\leq \|x_n - y\|^{-1} \inf_{y \in \mathcal{N}(\Phi)} \|x_n - y_n - \|x_n - y_n\|y\|
= \|x_n - y_n\|^{-1} \text{dist}(x_n, \mathcal{N}(\Phi)) \geq \frac{1}{2}. \quad \square$$

From Proposition 8, we obtain the following result.

Proposition 9. Let $K$ be a compact linear mapping of a normed linear space $V$ into itself. Let $\Phi = I - K$. Then $\mathcal{R}(\Phi)$ is a closed subspace of $V$.

Proof of Proposition 9. Let $\{x_m\}$ be a sequence in $V$ whose image $\Phi x_m$ converges to an element $y \in V$. To show that $\mathcal{R}(\Phi)$ is closed, we must show that

$$y = \Phi x \quad \text{for some } x \in V.$$ 

In fact, by Proposition 8, the sequence $\{d_n\}$ where $d_n = \text{dist}(x_n, \mathcal{N}(\Phi))$ is bounded. Choose as before a sequence $\{y_n\} \subset \mathcal{N}(\Phi)$ such that $d_n \leq \|x_n - y_n\| \leq 2d_n$. Write $w_n = x_n - y_n$,

we consequently have that the sequence $\{w_n\}$ is bounded while the sequence $\{\Phi w_n\}$ converges to $y$.

- Since $K$ is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{Kw_n\}$ converges to an element $w_0 \in V$.

- Thus, since $w_n = \Phi w_n + Kw_n$, the sequence $\{w_n\}$ converges to $y + w_0$.

By the continuity of $\Phi$, we have $\Phi(y + w_0) = y. \quad \square$

To proceed further, we need the following simple result.
Lemma 10 (Riesz). Let $V$ be a normed linear space and $M$ a proper closed subspace of $V$. Then, for any $\theta < 1$, there exists an element $x_\theta \in V$ satisfying $|x_\theta| = 1$ and $\text{dist}(x_\theta, M) \geq \theta$.

Proof of Lemma 10. Let $x \in V \setminus M$. Since $M$ is closed, we have
$$\text{dist}(x, M) = \inf_{y \in M} \|x - y\| = d > 0.$$ Consequently there exists an element $y_\theta \in M$ such that
$$\|x - y_\theta\| \leq \frac{d}{\theta},$$ so that, defining
$$x_\theta = \frac{x - y_\theta}{\|x - y_\theta\|},$$
we have $\|x_\theta\| = 1$ and for any $y \in M$,
$$\|x_\theta - y\| = \frac{x - y_\theta - \|y_\theta - x\|y}{\|y_\theta - x\|} \geq \frac{d}{\|y_\theta - x\|} \geq \theta. \quad \Box$$

From Proposition 9 and Lemma 10, we obtain the following.

Proposition 11. Let $K$ be a compact linear mapping of a normed linear space $V$ into itself.

(i) Let $R_j = \Phi^j(V)$, $j \in \mathbb{N}$. Then $\exists k \in \mathbb{N}$ such that $R_j = R_k$ for all $j \geq k$.

(ii) Let $N_j = \Phi^{-j}(0)$, $j \in \mathbb{N}$. Then $\exists \ell \in \mathbb{N}$ such that $N_j = N_\ell$ for all $j \geq \ell$.

Proof. (i) By Proposition 9, the sets $R_j$ form a non-increasing sequence of closed subspaces of $V$. Suppose no two of these spaces coincide. Then
$$R_j \not\subsetneq R_{j-1}, \quad j \geq 1.$$ Hence, by Lemma 10, $\exists$ a sequence $\{y_j\} \subset V$ such that $y_j \in R_j$, $\|y_j\| = 1$ and $\text{dist}(y_j, R_{j+1}) \geq \frac{1}{2}$. Thus, if $j > k$, $K y_k - K y_j = y_k + (-y_j - \Phi y_k - \Phi y_j)$
$$= y_k - y \quad \text{for some } y \in R_{k+1}.$$ Hence $\|K y_k - K y_j\| \geq \frac{1}{2}$, contrary to the compactness of $K$.

(ii) Let $N_j = \Phi^{-j}(0)$, $j \in \mathbb{N}$, which is closed by the continuity of $\Phi$.
- Then the sets $N_j$ form a non-decreasing sequence of closed subspaces of $V$.
- By applying an analogous argument based on Lemma 10 to that used in (i), we obtain $N_\ell = N_k$ for all $\ell \geq \text{some integer } \ell. \quad \Box$
Proof of Theorem 6. It is convenient to split the proof into four stages.

**Step 1. Claim:** If $\mathcal{N}(\Phi) = \emptyset$, then $\mathcal{R}(\Phi) = V$.

By Proposition 11 (i), for some $k \in \mathbb{N}$ and for all $y \in V$, $\Phi^k y \in \mathcal{R}_k = \mathcal{R}_{k+1}$ and so

$$\Phi^k y = \Phi^{k+1} x \quad \text{for some } x \in V.$$

Therefore

$$\Phi^k (y - \Phi x) = 0,$$

and so $y = \Phi x$, since $\Phi^{-k}(0) = \Phi^{-1}(0) = 0$. Consequently, $\mathcal{R}(\Phi) = \mathcal{R}_k = V$.

**Step 2. Claim:** If $\mathcal{R}(\Phi) = V$, then $\mathcal{R}(\Phi) = \emptyset$.

Let $\ell \in \mathbb{N}$ be as indicated in Proposition 11 (ii). If $\mathcal{R}(\Phi) = V$, then

$$\forall y \in \mathcal{N}_\ell, \; \exists x \in V \text{ such that } y = \Phi^\ell x.$$

Consequently, $\Phi^{2\ell} x = 0$, so that $x \in \mathcal{N}_{2\ell} = \mathcal{N}_\ell$, whence $y = \Phi^\ell x = 0$.

**Step 3.** In case (ii), the boundedness of the operator $\Phi^{-1} = (I - K)^{-1}$ follows from Proposition 7. □

V. The Adjoint of a Bounded Operator

The concept of adjoint operator extends the notion of transpose of an $m \times n$ matrix $A$ and plays a crucial role in determining compatibility conditions for the solvability of several problems.

The transpose $A^\perp$ is characterized by the identity

$$\langle Ax, y \rangle_{\mathbb{R}^m} = \langle x, A^\perp y \rangle_{\mathbb{R}^n}.$$

We extend precisely this relation to define the adjoint of a bounded linear operator.

- Let $L \in \mathcal{L}(H_1, H_2)$. If $y \in H_2$ is fixed, the real-valued map

$$T_y : x \mapsto \langle Lx, y \rangle_{H_2}$$

defines an element of $H_1^*$. In fact

$$|T_y x| = |\langle Lx, y \rangle| \leq \|Lx\|_{H_2} \|y\|_{H_2} \leq \|L\|_{\mathcal{L}(H_1, H_2)} \|y\|_{H_2} \|x\|_{H_1}$$

so that $\|T_y\| \leq \|L\|_{\mathcal{L}(H_1, H_2)} \|y\|_{H_2}$.

- From Riesz’s theorem, there exists a unique $w \in H_1$ depending on $y$, which we denote by $w = L^* y$, such that

$$T_y x = \langle x, L^* y \rangle_{H_1}, \quad \forall x \in H_1, \; \forall y \in H_2.$$

This defines $L^*$ as an operator from $H_2$ into $H_1$, which is called the adjoint of $L$. Precisely:
**Definition.** The operator $L^* : H_2 \to H_1$ defined by the identity

$$\langle Lx, y \rangle_{H_2} = \langle x, L^* y \rangle_{H_1}, \quad \forall x \in H_1, \quad \forall y \in H_2$$

is called the adjoint of $L$.

- The following properties are immediate consequences of the definition of adjoint.

**Proposition 12.** Let $L, L_1 \in \mathcal{L}(H_1, H_2)$ and $L_2 \in \mathcal{L}(H_2, H_3)$. Then
  (a) $L^* \in \mathcal{L}(H_2, H_1)$. Moreover, $L^{**} = L$ and
  $$\|L^*\|_{\mathcal{L}(H_2, H_1)} = \|L\|_{\mathcal{L}(H_1, H_2)}.$$  

  (b) $(L_2 L_1)^* = L_1^* L_2^*$. In particular, if $L$ is an isomorphism, then
  $$L^{-1})^* = (L^*)^{-1}.$$  

- The following proposition is useful.

**Proposition 13.** Let $L : H_1 \to H_2$ be compact. Then $L^* : H_2 \to H_1$ is compact.

**Proof.** We use Proposition 5. Let $\{x_k\} \subset H_2$ and $x_k$ converges to 0 weakly.

**Claim:** $\|L^* x_k\| \to 0$ strongly in $H_2$. Indeed, since $L \in \mathcal{L}(H_2, H_1)$, we have

$$L^* x_k \to 0 \text{ weakly in } H_1.$$ 

Thus, the compactness of $L$ and Proposition 5 entails $L L^* x_k \to 0$ strongly in $H_2$. Then, since the weak convergence of $x_k$ yields $\|x_k\| \leq M$ for some constant $M$,

$$\|L^* x_k\|^2_{H_2} = \langle L^* x_k, L^* x_k \rangle_{H_1} = \langle x_k, L L^* x_k \rangle_{H_2}$$

$$\leq \|x_k\| \|L L^* x_k\|_{H_2} \leq M \|L L^* x_k\|_{H_2} \to 0. \quad \Box$$

- The next theorem extends relations well known in the finite-dimensional case.

**Theorem 14.** Let $L \in \mathcal{L}(H_2, H_1)$. Then
  (a) $\mathcal{R}(L) = \mathcal{N}(L^*)^\perp$.
  (b) $\mathcal{N}(L) = \mathcal{R}(L^*)^\perp$.

**Proof.** (a) (i) Let $z \in \mathcal{R}(L)$. Then, $\exists x \in H_1$ such that $z = Lx$ and, if $y \in \mathcal{N}(L^*)$, we have

$$\langle z, y \rangle_{H_2} = \langle Lx, y \rangle_{H_2} = \langle x, L^* y \rangle_{H_1} = 0.$$

Thus, $\mathcal{R}(L) \subseteq \mathcal{N}(L^*)^\perp$. Since $\mathcal{N}(L^*)^\perp$ is closed, it follows that

$$\overline{\mathcal{R}(L)} \subseteq \mathcal{N}(L^*)^\perp.$$ (ii) On the other hand, if $z \in \mathcal{R}(L)^{\perp}$, for every $x \in H_1$, we have

$$0 = \langle Lx, z \rangle_{H_2} = \langle x, L^* z \rangle_{H_1}$$
whence $L^*z = 0$. Therefore
\[ \mathcal{R}(L)\perp \subseteq \mathcal{N}(L^*) , \]
which is equivalent to
\[ \mathcal{N}(L^*)\perp \subseteq \overline{\mathcal{R}(L)} . \]
(b) Substituting $L^*$ for $L$ in (a), we deduce
\[ \overline{\mathcal{R}(L^*)} = \mathcal{N}(L) \]
which is equivalent to $\mathcal{N}(L) = \mathcal{R}(L^*)\perp$. \qed

VI. Fredholm’s Alternative in Hilbert Spaces.

We introduce some terminology.

**Definition.** Let $V_1, V_2$ be Hilbert spaces and $\Phi : V_1 \to V_2$. Then $\Phi$ is said to be a *Fredholm operator* if $\mathcal{N}(\Phi)$ and $\mathcal{R}(\Phi)\perp$ have finite dimensions.

The **index** of $\Phi$ is the integer
\[ \text{ind } \Phi = \dim \mathcal{N}(\Phi) - \dim \mathcal{R}(\Phi)\perp = \dim \mathcal{N}(\Phi) - \dim \mathcal{N}(\Phi^*) . \]

We have:

**Theorem 15 (Fredholm’s Alternative).** Let $V$ be a Hilbert space and $K \in \mathcal{L}(V)$ be a compact operator. Then $\Phi : I - K$ is a Fredholm operator with zero index. Moreover, $\Phi^* = I - K^*$,
\[ (5) \quad \mathcal{R}(\Phi) = \mathcal{N}(\Phi^*)\perp \]
and
\[ (6) \quad \mathcal{N}(\Phi) = \{0\} \iff \mathcal{R}(\Phi) = V . \]

The last formula shows that $\Phi$ is one-to-one iff it is onto.

In other words, uniqueness for the equation
\[ (7) \quad x - Kx = f \]
is equivalent to existence for every $f \in V$.

- The same thing holds for the adjoint $\Phi^* = I - K^*$ and the associated equation
  \[ y - K^*y = g . \]
- Let $d = \dim \mathcal{R}(\Phi)\perp = \dim \mathcal{N}(\Phi^*) > 0$.
  Then (5) says that the equation (7) is solvable iff $f \perp \mathcal{N}(\Phi^*)$, that is, iff $\langle f, y \rangle = 0$, for every $y$ of
  \[ y - K^*y = 0 . \]
If \( y_1, \ldots, y_n \) span \( \mathcal{N}(\Phi^*) \), this amounts to asking that the \( d \) compatibility relations

\[
\langle f, y_j \rangle = 0, \quad j = 1, \ldots, n
\]

are necessary and sufficient conditions for the solvability of (7).

To prove Theorem 15, it remains to show that (i) \( \dim \mathcal{N}(\Phi) < \infty \) and (ii) \( \dim \mathcal{N}(\Phi) = \dim \mathcal{N}(\Phi^*) \).

(i) If (i) were not correct, there would exist an orthonormal system \( \{ \varphi_i \} \) satisfying

\[
0 = \Phi \varphi_i = \varphi_i - K \varphi_i, \quad i \in \mathbb{N}.
\]

Since the operator \( K \) is compact, we can select a strongly convergent subsequence \( \{ \varphi_{i_j} \} \) of \( \{ \varphi_i \} \) in \( V \). This contradicts the statement \( \| \varphi_i - \varphi_j \| = \sqrt{2}, \forall i, j \in \mathbb{N} \) with \( i \neq j \).

(ii) To prove (ii), we assume w.l.o.g. \( d = \dim \mathcal{N}(\Phi) \leq \dim \mathcal{N}(\Phi^*) = d^* < \infty \), for otherwise we could replace \( \Phi \) by \( \Phi^* \) and \( \Phi^* \) by \( \Phi^{**} \).

Consider the orthonormal basis \( \{ \varphi_1, \ldots, \varphi_d \} \) of \( \mathcal{N}(\Phi) \) and the orthonormal basis \( \{ \psi_1, \ldots, \psi_d \} \) of \( \mathcal{N}(\Phi^*) \). We consider the Fredholm operator

\[
Sx = \Phi x - \sum_{i=1}^{d} \langle \varphi_i, x \rangle \psi_i, \quad x \in V.
\]

On account of Theorem 14 (a), the null space of the operator \( S \) satisfies

\[
\mathcal{N}(S) = \{ x \in V : Sx = 0 \} = \{ 0 \}.
\]

Theorem 6 then implies that the mapping \( S : H \to H \) is surjective. Consequently \( d = d^* \), i.e. \( \dim \mathcal{N}(\Phi) = \mathcal{N}(\Phi^*) \).

II.3. Solvability for Abstract Variational Problem

- Let us go back to the variational problem

\[
a(u, v) = \langle F, v \rangle, \quad \forall v \in V,
\]

and suppose that Lax-Milgram cannot be applied, since, for instance, \( a \) in not \( V \)-coercive.

- The problem involves two Hilbert spaces:

  \( V \), the space where we seek the solution, and

  \( V^* \), which the data \( F \) belongs to.

  Let us introduce a third space \( H \), intermediate between \( V \) and \( V^* \).

- In boundary value problem, usually \( H = L^2(\Omega) \), with \( \Omega \) bounded domain in \( \mathbb{R}^n \),

  while \( V \) is a Sobolev space.

- In practice, we often meet a pair of Hilbert spaces \( V, H \) with the following properties:

  (i) \( V \hookrightarrow H \), i.e. \( V \) is continuously embedded in \( H \). Recall that this simply means that the identity operator \( I_{V \to H} \), from \( V \) to \( H \), is continuous or, equivalently that there exists \( C \) such that

\[
\| u \|_H \leq C \| u \|_V, \quad \forall u \in V.
\]
(ii) \( V \) is dense in \( H \).

- Using Riesz’s theorem, we may identify \( H \) with \( H^* \).
- Also, we may \textbf{continuously embed} \( H \) into \( V^* \), so that any element in \( H \) can be regarded as an element of \( V^* \). Indeed, for any fixed \( u \in H \), the functional \( T_u \) defined by

\[
\langle T_u, v \rangle_* = \langle u, v \rangle_H, \quad \forall v \in V
\]

is continuous in \( V \), since the Schwartz inequality and (8) give

\[
|\langle u, v \rangle_H| \leq \|u\|_H \|v\|_H \leq C \|u\|_H \|v\|_V.
\]

Then we have a continuous map \( u \mapsto T_u \) from \( H \) into \( V^* \) with \( \|T_u\|_{V^*} \leq C \|u\|_H \).
- Moreover, the map \( u \mapsto T_u \) is one-to-one; indeed, if \( T_u = 0 \), then

\[
\langle u, v \rangle_H = 0, \quad \forall v \in V,
\]

which forces \( u = 0 \), using the fact that \( V \) is dense in \( H \).
- Thus the map \( u \mapsto T_u \) is a continuous embedding.

This allows the \textit{identification} of \( u \) with an element of \( V^* \), which means that, instead of (9), we can write

\[
\langle u, v \rangle_* = \langle u, v \rangle_H, \quad \forall v \in V,
\]

regarding \( u \) on the left as an element of \( V^* \) and on the right as an element of \( H \).
- Finally, it can be shown that \( V \) and \( H \) are dense in \( V^* \).

Thus, we have \( V \hookrightarrow H \hookrightarrow V^* \) with \textbf{dense embeddings}. We call \((V, H, V^*)\) a \textbf{Hilbert triplet}.

\textbullet \ To state the main result we introduce weakly coercive forms and their adjoints.

\textbf{Definition.} \textit{The bilinear form} \( a(u, v) \) \textit{is said to be weakly coercive with respect to the pair} \((V, H)\) \textit{if there exist} \( \lambda_0 \in \mathbb{R} \) \textit{and} \( \alpha > 0 \) \textit{such that}

\[
a(u, v) + \lambda_0 \|v\|^2 \geq \alpha \|u\|^2_V, \quad \forall v \in V.
\]

\textbf{Definition.} \textit{The adjoint forms} \( a_* \) \textit{of} \( a \) \textit{is given by}

\[
a_*(u, v) = a(v, u).
\]

Set

\[
\mathcal{N}(a) = \{ u : a(u, v) = 0 \ \forall v \in V \}, \quad \mathcal{N}(a_*) = \{ w : a_*(w, v) = 0 \ \forall v \in V \}.
\]
Theorem 16. Let \((V, H, V^*)\) be a Hilbert triplet, with \(V\) compactly embedded in \(H\). Let \(F \in V^*\) and \(a\) be a bilinear form in \(V\), continuous and weakly coercive with respect to \((V, H)\). Then

(a) Either the equation

\[
a(u, v) = \langle F, v \rangle_*, \quad \forall v \in V
\]

has a unique solution \(\bar{u}\) and \(\|\bar{u}\| \leq C\|F\|_{V^*},\)

(b) or

\[
\dim N(a) = \dim N(a_*) = d < \infty
\]

and (10) is solvable iff \(\langle F, v \rangle_* = 0\) for every \(w \in N(a_*)\).

Some comments are in order. The following dichotomy holds either (10) has a unique solution for every \(F \in V^*\), or the homogeneous equation \(a(u, v) = 0\) has nontrivial solutions.

The same conclusion holds for the adjoint equation

\[
a^*(u, v) = \langle F, v \rangle_*, \quad \forall v \in V.
\]

If \(w_1, \cdots, w_d\) span \(N(a_*)\), (10) is solved iff the \(d\) compatibility conditions

\[
\langle F, w_j \rangle_*, \quad j = 1, \cdots, d,
\]

holds. In this case, equation (10) has infinitely many solutions given by

\[
u = \bar{u} + \sum_{j=1}^d c_j z_j,
\]

where \(\bar{u}\) is a particular solution of (10), \(z_1, \cdots, z_d\) span \(N(a)\) and \(c_1, \cdots, c_d\) are arbitrary constants.

Proof of Theorem 16. The strategy is to write equation

\[
a(u, x) = \langle F, v \rangle_*
\]

in the form

\[
(I_V - K)u = g,
\]

where \(I_V\) is the identity operator in \(V\) and \(K : V \to V\) is compact.

\(\odot\) Let \(J : V \to V^*\), the embedding of \(V\) into \(V^*\).

\(\odot\) Recall that \(J\) is the composition of the embedding \(I_{V \to H}\) and \(I_{H \to V^*}\).

\(\odot\) Since \(I_{V \to H}\) is compact and \(I_{H \to V^*}\) is continuous, we infer from Proposition 6 that \(J\) is compact.

\(\odot\) We write (11) in the form

\[
a_{\lambda_0}(u, v) = a(u, v) + \lambda_0 \langle u, v \rangle_H = \langle \lambda_0 J u + F, v \rangle_*,
\]

where \(\lambda_0 > 0\) is such that \(a_{\lambda_0}\) is coercive.
Since, for each fixed $u \in V$, the linear map

$$v \mapsto a_{\lambda_0}(u, v)$$

is continuous in $V$, there exists $L \in \mathcal{L}(V, V^*)$ such that

$$\langle Lu, v \rangle^* = a_{\lambda_0}(u, v), \quad \forall u, v \in V.$$  

Thus, the equation (11) is equivalent to

$$\langle Lu, v \rangle^* = (\lambda_0Ju + F, v)^*, \quad \forall v \in V,$$

and therefore to

(12) \hspace{1cm} Lu = \lambda_0Ju + F.

Since $a_{\lambda_0}$ is $V$-coercive, from the Lax-Milgram theorem, the operator $L$ is an isomorphism between $V$ and $V^*$ and (12) can be written in the form

$$u - \lambda_0L^{-1}Ju = L^{-1}F.$$  

Letting $g = L^{-1}F \in V$ and $K = \lambda_0L^{-1}J$, (12) becomes

$$(I_V - K)u = g,$$

where $K : V \to V$.

Since $J$ is compact and $L^{-1}$ is continuous, $K$ is compact.

Applying the Fredholm Alternative Theorem and rephrasing the conclusion in terms of bilinear forms we conclude the proof. □