Compactness, Fredholm Alternative and Spectrum

I. Compactness

Definition. *We say that $E \subseteq X$ is* **compact** *if from eavery open covering of E it is possible to extract a finite subcovering of E.*

Definition. *A subset E of a normed space X is* **sequentially compact** *if for every sequence $\{x_k\} \subset E$ there exists a subsequence $\{x_k\}$, converging in X.*

Theorem 1. *Let X be a normed space and $E \subset X$. Then E is compact iff it is sequentially compact.*

- While a compact set is always *closed and bounded*, the following example exhibits a closed and bounded set which is not compact in ℓ^2.

Example. Consider the real Hilbert space

$$\ell^2 = \{\mathbf{x} = \{x_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} x_k^2 < \infty,\, x_k \in \mathbb{R}\}$$

endowed with

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^{\infty} x_k y_k, \quad \text{and} \quad \|\mathbf{x}\|^2 = \sum_{k=1}^{\infty} x_k^2.$$

Let $\mathbf{E} = \{\mathbf{e}_k\}_{k \geq 1}$, where $\mathbf{e}^1 = \{1, 0, 0, \cdots\}$, $\mathbf{e}^2 = \{0, 1, 0, \cdots\}$, etc.
Observe that E constitute an orthonormal basis in ℓ^2. Thus, E is closed and bounded in ℓ^2. Howeve, E is **not sequentially compact**. Indeed, $\|\mathbf{e}_i - \mathbf{e}_j\| = \sqrt{2}$, if $j \neq k$, and therefore no subsequence of $\{\mathbf{e}^k\}_{k \geq 1}$ can be convergent.

- In fact, only in finite-dimensional spaces each closed and bounded set is compact.

Theorem 1. *Let B be a Banach space. B is finite-dimensional iff the unit ball $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\}$ is compact.*

II. Weak Convergence and Compactness.

Definition. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A sequence $\{x_k\} \subset H$ converges weakly to $x \in H$ and we write $x_k \rightharpoonup x$, if*

$$\langle F, x_k \rangle_* \to \langle F, x \rangle_*, \quad \forall F \in H^*.$$

- The convergence in norm is called **strong convergence**.
- From Riesz's representation theorem, it follows that $\{x_k\} \subset H$ **converges weakly** to H iff
$$\langle x_k, y \rangle \to \langle x, y \rangle, \quad \forall y \in H.$$
- **The weak limit is unique**, since $x_k \to x$ and $x_k \to z$ implv
$$\langle x - z, y \rangle, \quad \forall y \in H,$$
whence $x = z$.

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- **Strong convergence implies weak convergence**, since Schwarz's inequality gives
$$\|\langle x_k, y\rangle\| \le \|x_k - x\|\|y\|.$$

– The two notions of convergence are equivalent in the finite-dimensiona spaces.
– It is not so in infinite dimensions, as the following example shows.

Example. Let $H = L^2(0, 2\pi)$. The sequence $v_k = \cos kx$, $k \ge 1$, is **weakly converhent to zero**. In fact, $\forall f \in L^2(0, 2\pi)$, the Riemann-Lbesgue theorem on the Fourier coefficients of f implies that
$$\langle f, v_k\rangle = \int^{2\pi_0} f(x)\cos kx\, dx \to 0, \quad \text{as } k \to \infty.$$
However
$$\|u_k\|_{L^2(0,2\pi)} = \sqrt{\pi}$$
and therefore $\{v_k\}_{k\ge1}$ does not converge strongly.

Example. Let $\{\varphi\}_{i=1,2,\cdots}$ denote an orthonormal system in the Hilbert space $\mathcal{H}$, and we observe
$$\|\varphi_i - \varphi_j\| = \sqrt{2}, \quad \forall i, j \in \mathbb{N}, \quad i \ne j.$$
Consequently $\{\varphi_i\}_{i=1,2,\cdots}$ does not contains a strongly convergent subsequence. According to Bessel's inequality,
$$\sum_{i=1}^{\infty} |\langle \varphi, f\rangle|^2 \le \|f\|^2 < \infty, \quad \forall f \in \mathcal{H},$$
and we infer
$$\lim_{i\to\infty} \langle \varphi_i, f\rangle = 0 = \langle 0, f\rangle \quad \forall f \in \mathcal{H}.$$
We therefore obtain $\varphi \rightharpoonup 0$ as $i \to \infty$. Note that
$$\|0\| < 1 = \liminf_{k\to\infty} \|\varphi\|.$$

Principle of Uniform Boundedness. *On the Hilbert space H let the sequence of bounded functionals $A_n : H \to \mathbb{C}$ with $n \in \mathbb{N}$ be given, such that each element $f \in H$ possesses a constant $c_f \in [0, +\infty)$ with the property*

(1) $$|A_n f| \le c_f, \quad n = 1, 2, \cdots .$$

Then we have a constant $\alpha \in [0, \infty)$ satisfying
$$\|A_n\| \le \alpha, \quad \forall n \in \mathbb{N}.$$

Proof. **Step 1.** Choose $f_0 \in H$, $\varepsilon > 0$ and a constant $c \ge 0$. Let $A : H \to \mathbb{C}$ denote a bounded linear functional such that

(2) $$|Af| \le c \quad \forall f \in H \ \text{ with } \ \|f - f_0\| \le \varepsilon.$$

Then we have
$$\|A\| \le \frac{2c}{\varepsilon};$$
indeed, if $\|x\| \le 1$, $\exists f$ with $\|f - f_0\| \le \varepsilon$ such thar $x = \frac{1}{\varepsilon}(f - f_0)$, and we infer
$$|Ax| = \left|\frac{1}{\varepsilon}Af - \frac{1}{\varepsilon}Af_0\right| \le \frac{1}{\varepsilon}(|Af| + |Af_0|) \le \frac{2c}{\varepsilon}.$$

Step 2. If the statement (2) does not hold true, Step 1 together with the continuity of the functional $\{A_n\}$ enables us to construct a sequence of balls

$$\Sigma_n := \{f \in H : \|f - f_n\| \leq \varepsilon_n\}, \quad n \in \mathbb{N}$$

satisfying $\Sigma_1 \supset \Sigma_2 \supset \cdots$ with $\varepsilon_n \searrow 0$ as $n \to \infty$ and an index-sequence $1 < n_1 < n_2 < \cdots$ such that

$$|A_{n_j} x| \geq j, \quad \forall x \in \Sigma_j, \quad j \in \mathbb{N},$$

which yields a contradiction to (1). □

- We have observed that the norm in a Hilbert space is strongly continuous. With respect to weak convergence, the norm is only **lower semincontinuous**, as poperty (b) in the following theorem shows.

Theorem 2. *Let $\{x_k\} \subset H$ such that x_k weakly converges to x. Then*
(a) $\{x_k\}$ is bounded,
(b) $\|x\| \leq \liminf_{k\to\infty} \|x_k\|$.

Proof. (a) follows from the principle of uniform boundedness. For (b), it is enough to observe that

$$\|x\|^2 \leq \lim_{k\to\infty} \langle x_k, x\rangle \leq \|x\| \liminf_{k\to\infty} \|x_k\|. \qquad \square$$

- The usefulness of weak convegence is revealed by the following **compactness result**.

Theorem 3. *Every bounded sequence in a Hilbert space H contains a subsequence which is weakly convergent to an element $x \in H$.*

Proof. **Step 1.** The sequence $\{x_j\}$ is bounded and thus we have a constant $c \in [0, +\infty)$ such that $\|x_j\| \leq c, \quad j = 1, 2, \cdots$. On account of

$$|\langle x_1, x_j\rangle| \leq c\|x_1\|, \quad \forall j \in \mathbb{N},$$

we find a subsequence $\{x_j^{(1)}\} \subset \{x_j\}$ such that $\lim_{j\to\infty}\langle x_1, x_j^{(1)}\rangle$ exists.

- Noting that

$$|\langle x_2, x_j^{(1)}\rangle| \leq c\|x_2\|, \quad \forall j \in \mathbb{N},$$

  we select a further subsequence $\{x_j^{(2)}\} \subset \{x_j^{(1)}\}$ such that $\lim_{j\to\infty}\langle x_2, x_j^{(2)}\rangle$ exists.
- The continuation of this process gives a chain of subsequence

$$\{x_j\} \supset \{x_j^{(1)}\} \supset \{x_j^{(2)}\} \supset \cdots \supset \{x_j^{(k)}\}$$

  such that

$$\lim_{j\to\infty} \langle x_i, x_j^{(k)}\rangle \quad \text{exists for } i = 1, \cdots k.$$

- The Cantor's diagonal procedure then gives us the sequence $x_k^{(k)}$ such that

$$\lim_{k\to\infty} \langle x_i, x_k^{(k)}\rangle \quad \text{exists for all } i \in \mathbb{N}.$$

Step 2. Denote the linear subspace of all fnite linear combinations of x_j by $\mathcal{M}$, namely

$$\mathcal{M} = \{x : x = \sum_{i=1}^{N(x)} \alpha_i x_i, \quad \alpha_i \in \mathbb{C}, \quad N(x) \in \mathbb{N}\}.$$

Then

$$\lim_{k\to\infty} \langle x, x_k^{(k)} \rangle \quad \text{exists for all } x \in \mathcal{M}.$$

Making the transition to the linear closed subspace $\mathcal{M} \subset \overline{\mathcal{M}} \subset H$,

$$\lim_{k\to\infty} \langle y, x_k^{(k)} \rangle \quad \text{exists for all } y \in \overline{\mathcal{M}}.$$

Step 3. Due the projection theorem, each element $y \in H$ can be represented in the form

$$y = y_1 + y_2, , \quad \text{with } y_1 \in \overline{\mathcal{M}}, \;\; y_1 \in \mathcal{M}^\perp.$$

This implies the existence of

$$\lim_{k\to\infty} \langle y, x_k^{(k)} \rangle = \lim_{k\to\infty} \langle y_1 + y_2, x_k^{(k)} \rangle = \lim_{k\to\infty} \langle y_1, x_k^{(k)} \rangle, \quad \forall y \in H.$$

Step 4. From Step 3, we may define a linear functional A in H by setting

$$Ay = \lim_{k\to\infty} \langle y, x_k^{(k)} \rangle, \quad \forall y \in H.$$

Consider the bounded linear functional

$$A_i(y) = \langle x_i^{(i)}, y \rangle, \quad y \in H.$$

The principle of uniform boundedness gives us a constant $C \in [0, \infty)$ such that

$$\|x_n^{(n)}\| = \|A_n\| \leq C, \quad \forall n \in \mathbb{N}.$$

This and Theorem 1(b) imply $\|A\| \leq C$. Thus A is a bounded linear functional and Riesz representtion theorem then gives us an element $x \in H$ satisfying

$$A(y) = \langle x, y \rangle, \quad \forall y \in H.$$

We then obtain

$$\lim_{k\to\infty} \langle y, x_k^{(k)} \rangle = \lim_{k\to\infty} A_k(y) = A(y) = \langle x, y \rangle, \quad \forall y \in H,$$

which means that the sequence $\{x_k^{(k)}\}$ converges weakly. $\square$

III. Compact Operators

- Let V_1 and V_2 be normed linear spaces. Every operator in $\mathcal{L}(V_1, V_2)$ transforms bounded sets in V_1 into bounded sets in V_2.
  The subclass of operators that transform **bounded** sets into **pre-compact** sets is particularly important.

Definition. *A subset Σ of a normed linear space V is called* **precompact***, if each sequence $\{y_m\} \subset \Sigma$ contains a stronly convegent subsequence $\{y_{m_k}\}$, which means*

$$\lim_{k,\ell\to\infty} \|y_{m_k} - y_{n_\ell}\| = 0.$$

Definition. *Let V_1 and V_2 be normed linear spaces and $L \in \mathcal{L}(V_1, V_2)$. L is said to be* **compact** *if for every bounded subset E of V_1, the image $L(E)$ is pre-compact in V_2. This means that each sequence $\{x_m\}$ in V_1 with $\|x_m\|_{V_1} \le r$, for some $r \in \mathbb{R}$ and all $n \in \mathbb{N}$, contains a subsequence $\{x_{m_k}\}$ such that $\{Lx_{m_k}\} \subset V_2$ converges strongly.*

Proposition 4. *A compact operator $K : V_1 \to V_2$ is bounded.*

Proof. If $K : V_1 \to V_2$ were not bounded, there would exist a sequence $\{x_n\} \subset V_1$ with $\|x_m\| = 1$, for all $n \in \mathbb{N}$ and $\|Kx_m\|_2 \to \infty$ as $n \to \infty$. Therefore, we cannot select a convergent subsequence from $\{Kx_n\}$ in V_2. □

- An equivalent characterization of compact operators between Hilbert spaces may be given in terms of weak convergence.
  Indeed, an operator is compact iff it **converts weak convergence to strong convergence**. Precisely, we have the following.

Proposition 5. *Let H_1 and H_2 be Hilbert spaces and $L \in \mathcal{L}(H_1, H_2)$. L is compact iff for every sequence $\{x_k\} \subset H_1$,*

$$x_k \rightharpoonup x \ \text{ in } H_1 \text{ weakly} \quad \text{implies} \quad Lx_k \to Lx \ \text{ in } H_2 \text{ strongy.}$$

Proof. (⇒) Let $\{x_m\} \subset H_1$ be a sequence with $\|x_m\| \le 1$, $m \in \mathbb{N}$. According to Theorem 3, we have a subsequence $\{x_{m_k}\} \subset \{x_m\}$ which converges weakly to x in H_1 as $k \to \infty$.
Then, by assumption

$$Kx_{m_k} \to Kx \ \text{ strongly}, \quad k \to \infty.$$

Consequently, the operator $K : H_1 \to H_2$ is compact.
(⇐) Now let K be compact and $\{x_m\}$ denote a sequence weakly converges to x. We then have to prove $Kx_m \to Kx$ strongly as $m \to \infty$ in H_2.
If the latter statement were false, there would exist a number $d > 0$ and a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ satisfying

$$\|Kx_{m_k} - Kx\| \ge d > 0, \quad \forall n \in \mathbb{N}. \tag{3}$$

- On the other hand, by Theorem 2(b) the sequence $\{x_m\}$ is bounded in H_1. Since the operator is compact, we have a further subsequence

$$\{\widetilde{x}_{m_k}\} \subset \{x_{m_k}\} \quad \text{with} \quad K\widetilde{x}_{m_k} \to y \ \text{ strongly, as } n \to \infty.$$

- Moreover, $\{x_m\}$ converges weakly to Kx, which implies $y = Kx$. Hence $\|Kx_{m_k} - Kx\| \to 0$ as $m_k \to \infty$, in contradiction with (3). □

Example. From Theorem 3, the identity operator $I : H \to H$ is compact iff $\dim H < \infty$.
Also, any bounded operator with finite dimensional range is compact.

- The following proposition is useful.

Proposition 6. *Let $L : H_1 \to H_2$ be compact. If $G \in \mathcal{L}(H_2, H_3)$ or $G \in \mathcal{L}(H_0, H_1)$, then the operator $G \circ L$ or $L \circ G$ is compact.*

Proof. Since L is compact, the sequence $x_k \rightharpoonup x$ weakly in H_1 as $k \to \infty$ is transformed into the sequence Lx_k which converges strongly to Lx in H_2 as $k \to \infty$.
(i) If $G : H_2 \to H_3$ is continuous, we infer

$$G \circ Lx_k \to G \circ Lx \quad \text{as } n \to \infty \text{ in } H_3.$$

(ii) If $G : H_0 \to H_1$ is continuous, the sequence $y_k \rightharpoonup y$ weakly in H_0 as $k \to \infty$ is transformed into the sequence Gy_k which converges weakly to Gy in H_1 as $n \to \infty$. Since L is compact, the sequence $L \circ Gy_k$ converges strongly to $L \circ Gy$ in H_2 as $k \to \infty$. □

IV. The Fredholm Alternative

The Fredholm alternative concerns compact linear operators from a space V into itself and is an extension of the theory of linear mappings in finite dimensional spaces.

- The result we are going to present are extensions of well known facts concerning the solvability of linear algebraic systems of the form

  $$\mathbf{Ax} = \mathbf{b}, \tag{4}$$

  where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$.
- The following dichotomy holds: *either (4) has a unique solution for every* $\mathbf{b}$ *or the homogeneous equation* $\mathbf{Ax} = 0$ *has nontrivial solutions.*

- More precisely, system (4) is solvable iff $\mathbf{b}$ belong to the *column space* of $\mathbf{A}$, which is the orthogonal complement of $\ker(\mathbf{A}^\top)$.
- If $\mathbf{w}_1, \mathbf{w}_s$ span $\ker(\mathbf{A}^\top)$, this amounts to asking the s compatibility conditions, $0 \le s \le n$,

  $$\mathbf{b} \cdot \mathbf{w}_j = 0, \quad j = 1, \cdots, s.$$

  Finally, $\ker(\mathbf{A})$ and $\ker(\mathbf{A}^\top)$ have the same dimension and if $\mathbf{v}_1, \cdots, \mathbf{v}_n$ span $\ker(\mathbf{A})$, the general solution of (4) is given by

  $$\mathbf{x} = \overline{\mathbf{x}} + \sum_{j=1}^{s} c_j \mathbf{v}_j$$

  where $\overline{\mathbf{x}}$ is a particular solution of (4) and $c_1, \cdots, c_s$ are arbitrary constants.

- The extension to infinite-dimensional spaces requires some care.

Theorem 7. *Let K be a compact linear mapping of a normed linear space V into itself. Then either*
(i) the homogeneous equation $x - Kx = 0$ has a nontrivial solution $x \in V$ or
(ii) for each $y \in V$ the equation $x - Kx = y$ has a unique solution $x \in V$.
Furthermore, in case (ii), the operator $(I - K)^{-1}$ whose existence is asserted there is also bounded.

To prove Theorem 7, we first establish the following.

Proposition 8. *Let K be a compact linear mapping of a normed linear space V into itself. Let $\Phi = I - K$. Then $\exists$ a constant C such that*

$$\mathit{dist}\,(x, \mathcal{N}(\Phi)) \leq C\|\Phi x\|, \quad \forall v \in V.$$

Proof. Indeed, suppose that the result is not true. Then there exists a sequence $\{x_n\} \subset V$ satisfying $\|\Phi x_n\| = 1$ and $d_n = \text{dist}(x_n, \mathcal{N}(\Phi)) \to \infty$.

- Choose a sequence $\{y_n\} \subset \mathcal{N}(\Phi)$ such that $d_n \leq \|x_n - y_n\| \leq 2d_n$. Then if

$$z_n = \frac{x_n - y_n}{\|x_n - y_n\|}$$

  we have $\|z_n\| = 1$ and $\|\Phi z_n\| \leq d_n^{-1} \to 0$, so that the sequence $\{\Phi z_n\}$ converges to 0.
- But since K **is compact**, by passing to a subsequence if necessary, we may assume that the sequence $\{K z_n\}$ converges to an element $y_0 \in V$.
- Since $z_n = (\Phi + K)z_n$, we then also have $\{z_n\}$ converging to y_0 and hence $y_0 \in \mathcal{N}(\Phi)$.

$$\begin{aligned}\text{dist}\,(z_n, \mathcal{N}(\Phi)) = &\inf_{y \in \mathcal{N}(\Phi)} \|z_n - y\| \\ =&\|x_n - y\|^{-1} \inf_{y \in \mathcal{N}(\Phi)} \Big\| x_n - y_n - \|x_n - y_n\| y \Big\| \\ =&\|x_n - y_n\|^{-1} \text{dist}\,(x_n, \mathcal{N}(\Phi)) \geq \frac{1}{2}. \qquad \square\end{aligned}$$

From Proposition 8, we obtain the following result.

Proposition 9. *Let K be a compact linear mapping of a normed linear space V into itself. Let $\Phi = I - K$. Then $\mathcal{R}(\Phi)$ is a closed subspace of V.*

Proof of Proposition 9. Let $\{x_m\}$ be a sequence in V whose image Φx_m converges to an element $y \in V$. To show tht $\mathcal{R}(\Phi)$ is closed, we must **show that**

$$y = \Phi x \quad \text{for some } x \in V.$$

In fact, by Proposition 8, the sequence $\{d_n\}$ where $d_n = \text{dist}\,(x_n, \mathcal{N}(\Phi))$ is bounded. Choose as before a sequence $\{y_n\} \subset \mathcal{N}(\Phi)$ such that $d_n \leq \|x_n - y_n\| \leq 2d_n$. Write

$$w_n = x_n - y_n,$$

we consequently have that the sequence $\{w_n\}$ is bounded while the sequence $\{\Phi w_n\}$ conveges to y.

- Since K is compact, by passing to a subsequence if necessary, we may assume that the sequence $\{K w_n\}$ conveges to an element $w_0 \in V$.
- Thus, since $w_n = \Phi w_n + K w_n$, the sequence $\{w_n\}$ converges to $y + w_0$. By the continuity of Φ, we have $\Phi(y + w_0) = y$. $\square$

To proceeed further, we need the following simple result.

Lemma 10 (Riesz). *Let V be a normed linear space and M a proper closed subspace of V. Then, for any $\theta < 1$, there exists an element $x_\theta \in V$ satisfying $|x_\theta\| = 1$ and $\operatorname{dist}(x_\theta, M) \geq \theta$.*

Proof of Lemma 10. Let $x \in V \setminus M$. Since M is closed, we have

$$\operatorname{dist}(x, M) = \inf_{y \in M} \|x - y\| = d > 0.$$

Consequently there exists an element $y_\theta \in M$ such that

$$\|x - y_\theta\| \leq \frac{d}{\theta},$$

so that, defining

$$x_\theta = \frac{x - y_\theta}{\|x - y_\theta\|},$$

we have $\|x_\theta\| = 1$ and for any $y \in M$,

$$\begin{aligned} \|x_\theta - y\| =& \frac{x - y_\theta - \|y_\theta - x\| y}{\|y_\theta - x\|} \\ \geq& \frac{d}{\|y_\theta - x\|} \geq \theta. \quad \square \end{aligned}$$

From Proposition 9 and Lemma 10, we obtain the following.

Proposition 11. *Let K be a compact linear mapping of a normed linear space V into itself.*
(i) Let $\mathcal{R}_j = \Phi^j(V)$, $j \in \mathbb{N}$. Then $\exists k \in \mathbb{N}$ such that $\mathcal{R}_j = \mathcal{R}_k$ for all $j \geq k$.
(ii) Let $\mathcal{N}_j = \Phi^{-j}(0)$, $j \in \mathbb{N}$. Then $\exists \ell \in \mathbb{N}$ such that $\mathcal{R}_j = \mathcal{R}_\ell$ for all $j \geq \ell$.

Proof. (i) By Proposition 9, the sets $\mathcal{R}_j$ form a non-increasing sequence of closed subspaces of V. Suppose no two of these spaces coincide. Then

$$\mathcal{R}_j \subsetneq \mathcal{R}_{j-1}, \quad j \geq 1.$$

Hence, by Lemma 10, $\exists$a sequence $\{y_j\} \subset V$ such that $y_j \in \mathcal{R}_j$, $\|y_j\| = 1$ and $\operatorname{dist}(y_j, \mathcal{R}_{j+1}) \geq \frac{1}{2}$. Thus, if $j > k$,

$$\begin{aligned} Ky_k - Ky_j =& y_k + (-y_j - \Phi y_k - \Phi y_j) \\ =& y_k - y \quad \text{for some } y \in \mathcal{R}_{k+1}. \end{aligned}$$

Hence $\|Ky_k - Ky_\ell\| \geq \frac{1}{2}$, contrary to the compactness of K.
(ii) Let $\mathcal{N}_j = \Phi^{-j}(0)$, $j \in \mathbb{N}$, which is closed by the continuity of Φ.
- Then the sets $\mathcal{N}_j$ form a non-decreasing sequence of closed subspaces of V.
- By applying an anologous argument based on Lemma 10 to that used in (i), we obtain $\mathcal{N}_\ell = \mathcal{N}_k$ for all $\ell \geq$ some integer ℓ. $\square$

Proof of Theorem 6. It is convenient to split the proof into four stages.

Step 1. Claim: If $\mathcal{N}(\Phi) = \emptyset$, then $\mathcal{R}(\Phi) = V$.

By Proposition 11 (i), for some $k \in \mathbb{N}$ and for all $y \in V$, $\Phi^k y \in \mathcal{R}_k = \mathcal{R}_{k+1}$ and so

$$\Phi^k y = \Phi^{k+1} x \quad \text{for some } x \in V.$$

Therefore

$$\Phi^k (y - \Phi x) = 0,$$

and so $y = \Phi x$, since $\Phi^{-k}(0) = \Phi^{-1}(0) = 0$. Consequently, $\mathcal{R}(\Phi) = \mathcal{R}_k = V$.

Step 2. Claim: If $\mathcal{R}(\Phi) = V$, then $\mathcal{R}(\Phi) = \emptyset$.

Let $\ell \in \mathbb{N}$ be as indicated in Proposition 11 (ii). If $\mathcal{R}(\Phi) = V$, then

$$\forall y \in \mathcal{N}_\ell, \;\; \exists x \in V \;\text{ such that }\; y = \Phi^\ell x.$$

Consequently, $\Phi^{2\ell} x = 0$, so that $x \in \mathcal{N}_{2\ell} = \mathcal{N}_\ell$, whence $y = \Phi^\ell x = 0$.

Step 3. In case (ii), the boundedness of the operator $\Phi^{-1} = (I - K)^{-1}$ follows from Proposition 7. □

V.. The Adjoint of a Bounded Operator

The concept of **adjoint operator** extends the notion of transpose of an $m \times n$ matrix $\mathbf{A}$ and plays a crucial role in determining compatibility conditions for the solvability of several problems.

The transpose $\mathbf{A}^\perp$ is characterized by the identity

$$\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^m} = \langle \mathbf{x}, \mathbf{A}^\perp \mathbf{y} \rangle_{\mathbb{R}^n}.$$

We extend precisely this relation to define the adjoint of a bounded linear operator.

- Let $L \in \mathcal{L}(H_1, H_2)$. If $y \in H_2$ is fixed, the real-valued map

  $$T_y : x \mapsto \langle Lx, y \rangle_{H_2}$$

  defines an element of H_1^*. In fact

  $$|T_y x| = |\langle Lx, y \rangle| \leq \|Lx\|_{H_2} \|y\|_{H_2} \leq \|L\|_{\mathcal{L}(H_1,H_2)} \|y\|_{H_2} \|x\|_{H_1}$$

  so that $\|T_y\| \leq \|L\|_{\mathcal{L}(H_1,H_2)} \|y\|_{H_2}$.

- From Riesz's theorem, there exists a unique $w \in H_1$ depending on y, which we denote by $w = L^* y$, such that

  $$T_y x = \langle x, L^* y \rangle_{H_1}, \quad \forall x \in H_1, \;\; \forall y \in H_2.$$

  This defines L^* as an operator from H_2 into H_1, which is called the **adjoint** of L. Precisely:

Definition. *The operator $L^* : H_2 \to H_1$ defined by the identity*

$$\langle Lx, y\rangle_{H_2} = \langle x, L^* y\rangle_{H_1}, \quad \forall x \in H_1, \quad \forall y \in H_2 \tag{5}$$

is called the **adjoint** *of L.*

- The following properties are immediate consequences of the definition of adjoint.

Proposition 12. *Let $L, L_1 \in \mathcal{L}(H_1, H_2)$ and $L_2 \in \mathcal{L}(H_2, H_3)$. Then*
(a) $L^* \in \mathcal{L}(H_2, H_1)$. Moreover, $L^{**} = L$ and

$$\|L^*\|_{\mathcal{L}(H_2,H_1)} = \|L\|_{\mathcal{L}(H_1,H_2)}.$$

(b) $(L_2L_1)^* = L_1^* L_2^*$. In particulat, if L is an isomorphism, then

$$(L^{-1})^* = (L^*)^{-1}.$$

- The following proposition is useful.

Proposition 13. *Let $L : H_1 \to H_2$ be compact. Then $L^* : H_2 \to H_1$ is compact.*

Proof. We use Proposition 5. Let $\{x_k\} \subset H_2$ and x_k converges to 0 weakly.
Claim: $\|L^* x_k|\| \to 0$ **strongly in** H_2. Indeed, since $L \in \mathcal{L}(H_2, H_1)$, we have

$$L^* x_k \rightharpoonup 0 \ \text{ weakly in } H_1.$$

Thus, the compactness of L and Proposition 5 entails $LL^* x_k \to 0$ strongly in H_2. Then, since the weak convergence of x_k yields $\|x_k\| \le M$ for some constant M,

$$\begin{aligned}\|L^* x_k\|_{H_2}^2 =& \langle L^* x_k, L^* x_k\rangle_{H_1} = \langle x_k, LL^* x_k\rangle_{H_2}\\ \le& \|x_k\|\|LL^* x_k\|_{H_2} \le M\|LL^* x_k\|_{H_2} \to 0. \quad \square\end{aligned}$$

- The next theorem extends relations well known in the finite-dimensional case.

Theorem 14. *Let $L \in \mathcal{L}(H_2, H_1)$. Then*
(a) $\overline{\mathcal{R}(L)} = \mathcal{N}(L^*)^\perp$.
(b) $\mathcal{N}(L) = \mathcal{R}(L^*)^\perp$.

Proof. (a) (i) Let $z \in \mathcal{R}(L)$. Then, $\exists x \in H_1$ such that $z = Lx$ and, if $y \in \mathcal{N}(L^*)$, we have

$$\langle z, y\rangle_{H_2} = \langle Lx, y\rangle_{H_2} = \langle x, L^* y\rangle_{H_1} = 0.$$

Thus, $\mathcal{R}(L) \subseteq \mathcal{N}(L^*)^\perp$. Since $\mathcal{N}(L^*)^\perp$ is closed, it follows that

$$\overline{\mathcal{R}(L)} \subseteq \mathcal{N}(L^*)^\perp.$$

(ii) On the other hand, if $z \in \mathcal{R}(L)^\perp$, for every $x \in H_1$, we have

$$0 = \langle Lx, z\rangle_{H_2} = \langle x, L^* z\rangle_{H_1}$$

whence $L^*z = 0$. Therefore

$$\mathcal{R}(L)^\perp \subseteq \mathcal{N}(L^*),$$

which is equivalent to

$$\mathcal{N}(L^*)^\perp \subseteq \overline{\mathcal{R}(L)}.$$

(b) Substituting L^* for L in (a), we deduce

$$\overline{\mathcal{R}(L^*)} = \mathcal{N}(L)^\perp$$

which is equivalent to $\mathcal{N}(L) = \mathcal{R}(L^*)^\perp$. □

VI. Fredholm's Alternative in Hilbert Spaces.

We introduce some terminology.

Definition. *Let V_1, V_2 be Hilbert spaces and $\Phi : V_1 \to V_2$. Then Φ is said to be a* **Fredholm operator** *if $\mathcal{N}(\Phi)$ and $\mathcal{R}(\Phi)^\perp$ have finite dimensions.*
The **index** *of Φ is the integer*

$$\operatorname{ind}\Phi = \dim \mathcal{N}(\Phi) - \dim \mathcal{R}(\Phi)^\perp = \dim \mathcal{N}(\Phi) - \dim \mathcal{N}(\Phi^*).$$

We have:

Theorem 15 (Fredholm's Alternative). *Let V be a Hilbert space and $K \in \mathcal{L}(V)$ be a compact operator. Then $\Phi : I - K$ is a Fredholm operator with zero index. Moreover, $\Phi^* = I - K^*$,*

$$\mathcal{R}(\Phi) = \mathcal{N}(\Phi^*)^\perp \tag{5}$$

and

$$\mathcal{N}(\Phi) = \{0\} \Longleftrightarrow \mathcal{R}(\Phi) = V. \tag{6}$$

The last formula shows that Φ is one-to-one iff it is onto.
In other words, uniqueness for the equation

$$x - Kx = f \tag{7}$$

is equivalent to existence for every $f \in V$.

- The same thing hlds for the adjoint $\Phi^* = I - K^*$ and the associated equation

$$y - K^*y = g.$$

- Let $d = \dim \mathcal{R}(\Phi)^\perp = \dim \mathcal{N}(\Phi^*) > 0$.
  Then (5) says that the equation (7) is solvable iff $f \perp \mathcal{N}(\Phi^*)$, that is, iff $\langle f, y\rangle = 0$, for every y of

$$y - K^*y = 0.$$

If $y_1, \cdots, y_n$ span $\mathcal{N}(\Phi^*)$, this amounts to asking that the d **compatibility relations**

$$\langle f, y_j \rangle = 0, \quad j = 1, \cdots, n$$

are necessary and sufficient conditions for the solvability of (7).

To prove Theorem 15, it remains **to show that (i) dim**$\mathcal{N}(\Phi) < \infty$ and (ii) $\dim \mathcal{N}(\Phi) = \dim \mathcal{N}(\phi^*)$.

(i) If (i) were not correct, there would exist an orthonormal system $\{\varphi_i\}$ satisfying

$$0 = \Phi\varphi_i = \varphi_i - K\varphi_i, \quad i \in \mathbb{N}.$$

Since the operator K is compact, we can select a strongly convergent subsequence $\{\varphi_{i_j}\}$ of $\{\varphi_i\}$ in V. This contradics the statement $\|\varphi_i - \varphi_j\| = \sqrt{2}$, $\forall i, j \in \mathbb{N}$ with $i \neq j$.

(ii) To prove (ii), we assume w.l.o.g. $d = \dim \mathcal{N}(\Phi) \leq \dim \mathcal{N}(\Phi^*) = d^* < \infty$, for otherwise we could replace Φ by Φ^* and Φ^* by Φ^{**}.
Consider the orthonormal basis $\{\varphi_1, \cdots, \varphi_d\}$ of $\mathcal{N}(\Phi)$ and the orthonormal basis $\{\psi_1, \cdots, \psi_{d^*}\}$ of $\mathcal{N}(\Phi^*)$. We consider the Fredholm operator

$$Sx = \Phi x - \sum_{i=1}^{d} \langle \varphi_i, x \rangle \psi_i, \quad x \in V.$$

On account of Theorem 14 (a), the null space of the operator S satisfies

$$\mathcal{N}(S) = \{x \in V : Sx = 0\} = \{0\}.$$

Theorem 6 then implies that the mapping $S : H \to H$ is surjective. Consequently $d = d^*$, i.e. $\dim \mathcal{N}(\Phi) = \mathcal{N}(\phi^*)$.

II.3. Solvability for Abstract Variational Problem

- Let us go back to the variational problem

$$a(u, v) = \langle F, v \rangle_*, \quad \forall v \in V,$$

and suppose that Lax-Milgram cannot be applied, since, for instance, a in not V-coercive.

- The problem involves two Hilbert spaces:
  V, the space where we seek the solution, and
  V^*, which the data F belongs to.
  Let us introduce a third space H, intermediate between V and V^*.
- In boundary value problem, usually $H = L^2(\Omega)$, with Ω bounded domain in $\mathbb{R}^n$, while V is a Sobolev space.
- In practice, we often meet a pair of Hilbert spaces V, H with the following properties:

(i) $V \hookrightarrow H$, i.e. V is **continuously embedded in** H. Recall that this simply means that the identity operator $I_{V \to H}$, from V to H, is continuous or, equivalently that there exists C such that

$$\|u\|_H \leq C\|u\|_V, \quad \forall u \in V. \tag{8}$$

(ii) V is dense in H.

⊙ Using Riesz's theorem, we may identify H with H^*.

⊙ Also, we may **continuously embed** H into V^*, so that any element in H can be regarded as an element of V^*. Indeed, for any fixed $u \in H$, the functional T_u defined by

$$\langle T_u, v\rangle_* = \langle u, v\rangle_H, \quad \forall v \in V \tag{9}$$

is continuous in V, since the Schwartz inequality and (8) give

$$|\langle u, v\rangle_H| \leq \|u\|_H\|v\|_H \leq C\|u\|_H\|v\|_V.$$

Then we have a continuous map $u \mapsto T_u$ from H into V^* with $\|Tu\|_{V^*} \leq C\|u\|_H$.

- Moreover, the map $u \mapsto T_u$ is one-to-one; indeed, if $T_u = 0$, then

$$\langle u, v\rangle_H = 0, \quad \forall v \in V,$$

which forces $u = 0$, using the fact that V is dense in H.

- Thus the map $u \mapsto T_u$ is a continuous embedding.
This allows the *identification* of u with an element of V^*, which means that, instead of (9), we can write

$$\langle u, v\rangle_* = \langle u, v\rangle_H, \quad \forall v \in V,$$

regarding u on the left as an element of V^* and on the right as an element of H.

⊙ Finally, it can be shown that V and H are dense in V^*.
Thus, we have

$$V \hookrightarrow H \hookrightarrow V^*$$

with **dense embeddigs**. We call (V, H, V^*) a **Hilbert triplet**.

- To state the main result we introduce weakly coercive forms and their adjoints.

Definition. *The bilinear form $a(u, v)$ is said to be* **weakly coercive** *with respect to the pair (V, H) if there exist $\lambda_0 \in \mathbb{R}$ and $\alpha > 0$ such that*

$$a(u, v) + \lambda_0\|v\|^2 \geq \alpha\|u\|_V^2, \quad \forall v \in V.$$

Definition. *The adjoint forms a_* of a is given by*

$$a_*(u, v) = a(v, u).$$

Set

$$\mathcal{N}(a) = \{u : \ a(u, v) = 0 \ \ \forall v \in V\}, \quad \mathcal{N}(a_*) = \{w : \ a_*(w, v) = 0 \ \ \forall v \in V\}.$$

Theorem 16. *Let (V, H, V^*) be a Hilbert triplet, with V compactly embedded in H. Let $F \in V^*$ and a be a bilinear fom in V, continuous and weakly coercive with respect to (V, H). Then*

(a) Either the equation

$$a(u,v) = \langle F, v\rangle_*, \quad \forall v \in V \tag{10}$$

has a unique solution $\overline{u}$ and $\|\overline{u}\| \le C\|F\|_{V^*}$,

(b) or

$$\dim \mathcal{N}(a) = \dim \mathcal{N}(a_*) = d < \infty$$

and (10) is solvable iff $\langle F, v\rangle_* = 0$ for every $w \in \mathcal{N}(a_*)$.

Some comments are in order. The following dichotomy holds either (10) has a unique solution for every $F \in V^*$, or the homogeneous equation $a(u,v) = 0$ has nontrivial solutins.
The same conclusion holds for the adjoint equation

$$a^*(u,v) = \langle F, v\rangle_*, \quad \forall v \in V.$$

If $w_1, \cdots, w_d$ span $\mathcal{N}(a_*)$, (10) is solved iff the d compatibility conditions

$$\langle F, w_j\rangle_*, \quad j = 1, \cdots, d,$$

holds. In this case, equation (10) has infinitely many solutions given by

$$u = \overline{u} + \sum_{j=1}^{d} c_j z_j,$$

where $\overline{u}$ is a particular solution of (10), $z_1, \cdots, z_d$ span $\mathcal{N}(a)$ and $c_1, \cdots, c_d$ are arbitrary constants.

Proof of Theorem 16. The strategy is to write equation

$$a(u,x) = \langle F, v\rangle_* \tag{11}$$

in the form

$$(I_V - K)u = g,$$

where I_V is the identity operator in V and $K : V \to V$ is compact.

⊙ Let $J : V \to V^*$, the embedding of V into V^*.
- Recall that J is the composition of the embedding $I_{V\to H}$ and $I_{H\to V^*}$.
- Since $I_{V\to H}$ is compact and $I_{H\to V^*}$ is continuous, we infer from Proposition 6 that J is **compact**.
- We write (11) in the form

$$a_{\lambda_0}(u,v) = a(u,v) + \lambda_0\langle u, v\rangle_H = \langle \lambda_0 Ju + F, v\rangle_*,$$

  where $\lambda_0 > 0$ is such that a_{λ_0} **is coercive**.

Since, for each fixed $u \in V$, the linear map

$$v \mapsto a_{\lambda_0}(u, v)$$

is continuous in V, there exists $L \in \mathcal{L}(V, V^*)$ such that

$$\langle Lu, v\rangle_* = a_{\lambda_0}(u, v), \quad \forall u, v \in V.$$

Thus, the equation (11) is equivalent to

$$\langle Lu, v\rangle_* = \langle \lambda_0 Ju + F, v\rangle_*, \qquad \forall v \in V,$$

and therefore to

$$Lu = \lambda_0 Ju + F. \tag{12}$$

Since a_{λ_0} is V-coercive, from the Lax-MIlgram theorem, the operator L is an isomorphism between V and V^* and (12) can be written in the form

$$u - \lambda_0 L^{-1} Ju = L^{-1} F.$$

Letting $g = L^{-1}F \in V$ and $K = \lambda_0 L^{-1} J$, (12) becomes

$$(I_V - K)u = g,$$

where $K : V \to V$.
⊙ Since J is compact and L^{-1} is continuous, K is compact.
Applying the Fredholm Alternative Theorem and rephrasing the conclusion in terms of bilinear forms we conclude the proof. □