Celestial Dynamics: Homework I

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due in class on Apr 8th, 2009

1. **Angular diameter distance:** Consider a bright point source moving at a speed of v_r in a circular Keplerian orbit of radius R. The distance of the object from us is D_A, which is $\gg R$. The orbital plane is edge-on from our view. The angle between the center of the orbit and the point source viewed from us is $\Delta\theta$.

   1.1 It can be expected that as the point source moves, its line of sight velocity v_{los} and $\Delta\theta$ vary periodically with time. Show that v_{los} varies linearly with $\Delta\theta$ as follows

   $$v_{los} \approx \frac{v_r D_A \Delta\theta}{R}. \tag{1}$$

   Note that v_{los} can be measured from the Doppler shift of the point source.

   1.2 What are the maximum values of $\Delta\theta$ and v_{los}?

   1.3 Show that the centripetal acceleration a of the point source can be estimated from the variation of v_{los} (i.e. dv_{los}/dt) when the point source moves to the vicinity of $\Delta\theta \approx 0$. In other words, a can be measured from v_{los} when $v_{los} \ll v_r$.

   1.4 After knowing v_r, a, and the maximum value of $\Delta\theta$, show that

   $$D_A = \frac{v_r^2}{a\Delta\theta_{max}}, \tag{2}$$

   where $\Delta\theta_{max}$ denotes the maximum value of $\Delta\theta$. Since the distance of the object D_A is determined from the angular size $\Delta\theta$, D_A has been referred to as the "angular diameter distance", or sometimes is called "geometric distance", a distance measure relying simply on geometry of a circular Keplerian motion.

   1.5 The above calculations have applied to the distance measures of maser point sources orbiting a super-massive black hole in a Keplerian disk at the center of a Seyfert 2 galaxy far away from us. The purpose is to measure the Hubble constant with unprecedented precision. In one particular case, $a = 3 \times 10^{-2}$ cm s^{-2}, $\Delta\theta_{max} = 4$ mas, and $v_r = 1130$ km/s. Based on these parameters, estimate D_A and the mass of the central super-massive black hole.

2. **Mars' orbit:** 400 years ago, Kepler published his 1st and 2nd laws based on the heliocentric model fitting to the observations of Mars' movement on the sky. We should do a homework problem about Mars' orbit in memory of the historical

milestone. Solve Prob. 2.2 in Murray & Dermott. Note that you can obtain the simple instructions to solve the problem on the book website, but you are required to present all detailed steps in your solutions.

3. **Mass function:** We have learnt in class about the description of the orbital motion of one star in a binary system with respect to a reference plane. In this problem, we consider the orbital motion of m_1 described in terms of a_1, e, Ω, ω, I, and f. According to the conventional definition of I in Astronomy (i.e. $I = 90°$ means that the orbital plane is edge on), the reference plane should be perpendicular to the line of sight. Like the figure shown in class, you may choose the x-axis pointing to the ascending node from the origin (i.e. the center of mass of the binary system) to work out the problem.

   3.1 Show that the line-of-sign velocity (also known as "radial velocity") is given by

   $$K\left[\cos(\omega + f) + e\cos\omega\right], \tag{3}$$

   where the amplitude

   $$K = \frac{2\pi a_1 \sin I}{P(1-e^2)^{1/2}}. \tag{4}$$

   3.2 Then show that

   $$\frac{(m_2 \sin I)^3}{(m_1+m_2)^2} = \frac{K^3 P (1-e^2)^{3/2}}{2\pi G}, \tag{5}$$

   where the left hand side is conventionally called the mass function and can be determined by the measurable quantities via the Doppler shift of m_1 on the right hand. Then estimate the amplitude of the radial velocity K of the Sun due to the gravitational tug of Jupiter. Can you run faster than that speed?

4. **Gravitational slingshot:** Small bodies (such as spacecrafts, asteroids, or comets) can be accelerated or decelerated by a gravitational encounter with a planet. The simplest explanation for this is to consider this so-called slingshot effect as an analogy of a 1-D collision problem: a small body colliding gravitationally with a planet behaves almost like a massless particle hitting a rigid wall. As a result, the velocity of the small body relative to the planet does not change the magnitude but just changes the sign before and after the "collision". As a result, a small body gains twice of the planet's velocity during a head-on collision. Of course, in reality the orbital encounter should be a 3-D phenomenon involving the impact parameter and the inclination, and the concept "head-on" in the case of gravitational pull should be interpreted as a pitcher winding up to throw a high-velocity strike. Nevertheless, the relative velocity during the encounter can be proved to be still a conserved quantity in 3-D. Show that

the relative velocity v between a small body and a planet during a close encounter can be related to the Tisserand parameter T as follows

$$v^2 \approx 3 - 2T. \tag{6}$$

5. **Roche radius of a non-synchronized secondary:** In class we have studied the effective potential of a binary system with synchronous rotation. Consider the same binary system consisting of a primary body of mass m_1 and a secondary body of mass m_2 ($m_2 < m_1$), and the separation of these two bodies is a. However, the spin rate of the secondary body Ω_s is not necessarily equal to the orbital frequency n. For simplicity, we consider the case that $\mathbf{\Omega_s}$ is parallel to $\mathbf{n}$. We shall adopt the reference frame in which the origin lies at the center of the mass of the secondary and the x-axis points to the center of the mass of the primary.

   5.1 Using the standard system of units in class to non-dimensionalize the equations (i.e. $G(m_1 + m_2) = 1$ and $a = 1$)[1], show that the effective potential (i.e. gravitational plus centrifugal potential) at some point P of the secondary body can be written as the following form:

   $$\chi = \mu_1 \left[\frac{1}{r_1} + \frac{q}{r_2} + \frac{\beta^2}{2} (1+q)\, d^2 - x \right], \tag{7}$$

   where $q \equiv m_2/m_1$ (or $\equiv \mu_2/\mu_1$ in terms of the standard system of units) is the mass ratio and $\beta \equiv \Omega_s/n$ measures the synchrony of the secondary. r_1 and r_2 are the distances from the primary and the secondary to the point P, respectively. d is the shortest distance between the rotation axis of the secondary and the point P.

   5.2 Show that at the Lagrangian 1 point, the following equation holds

   $$q\left(\frac{1}{r_2^2} - \beta^2 r_2\right) = \frac{1}{(1-r_2)^2} + \beta^2 r_2 - 1. \tag{8}$$

   5.3 If $r_2 << 1$ and $\beta^2 r_2^3 << 1$, the Roche radius of the secondary can be approximated to

   $$r_2 \approx \left(\frac{q}{\beta^2 + 2}\right)^{1/3}. \tag{9}$$

   Compare this expression with the Roche radius in the synchronized case, and then explain in terms of simple physics why $\beta < 1$ (> 1) increases (decreases) the Roche radius of the secondary.

[1] In other words, to resume the normal units of the equation, the potential χ should be multiplied by $G(m_1 + m_2)/a$ and all distances (i.e. r_1, r_2, d, and x) should be multiplied by a.

6. **Zero-velocity curves through L_3:** Solve Prob. 3.1 in Murray & Dermott. Note that there is an error in the last line of the problem: the angular separation should be $23.9°$ rather than $23.5°$.