Stellar Physics: Homework III

Instructor: Gu, Pin-Gao
due in class on Apr 18th, 2014

1. **Thermal ionization fraction:**

   1.1 Calculate the thermal ionization fraction $x \equiv n_e/n$ contributed from two alkali elements K and Na for a gas with $n = 10^{17}$ cm^{-3} at $T = 3500$K. Assume that the ions are much less than the neutrals for each element to simplify the Saha equation. The ionization potentials of K and Na are $\chi_K = 4.34$ eV and $\chi_{Na} = 5.14$ eV. Take their abundances $X_K = 10^{-7}$ and $X_{Na} = 10^{-5.5}$.

   1.2 Perform the same calculation for x contributed from H with $X_H = 0.74$. Which contribution to x is dominant, from alkali elements or from H? Is the assumption of low ionization fractions reasonable in this problem?

2. **Thermodynamical quantities influenced by thermal ionization:**[1]

   2.1 In class, I derive the mean molecular weight for a fully ionized ideal gas. Now consider that a gas is partially ionized with the ionization fraction given by x. Show that the mean molecular weight of the gas is modified to the form $\mu = (\rho/m_u n)/(1+x)$.

   2.2 For an idea gas, derive
$$c_p = \left(\frac{\partial u}{\partial T}\right)_P + \frac{R}{\mu}\delta \tag{1}$$
   from the first law of thermodynamics, where $\delta \equiv -(\partial \ln \rho/\partial \ln T)_P$.

   2.3 We now focus on a pure hydrogen ideal gas, which was studied in class as an example for the calculation of x due to thermal ionization. Use the Saha equation to show that
$$\delta = 1 + \frac{1}{2}x(1-x)\left(\frac{5}{2} + \frac{\chi_H}{kT}\right). \tag{2}$$

   2.4 Use the internal energy $u = \frac{3RT}{2\mu} + u_{ion}$ with the additional contribution $u_{ion} = x\chi_H/m_u$ due to ionization. Derive
$$c_p = \frac{5\mathcal{R}}{2\mu} + \frac{\mathcal{R}}{\mu}(5/2 + \chi_H/kT)^2 \frac{x(1-x)}{2}. \tag{3}$$

---

[1] You should notice that the thermodynamical quantities (μ, c_p, ∇_{ad}, etc.) approach their "familiar" values as $x \to 0$ when working out the problems.

2.5 It can be shown that $\nabla_{ad} = P\delta/T\rho c_p$ (e.g. see "Stellar Interiors" by Hansen & Kawaler). Use this expression to show that

$$\nabla_{ad} = \frac{2 + x(1-x)(5/2 + \chi_H/kT)}{5 + x(1-x)(5/2 + \chi_H/kT)^2}. \tag{4}$$

Is this value larger or smaller than ∇_{ad} for a monatomic (i.e. non-ionized) ideal gas?

3. **Polytrope:**

3.1 Show that the gravitational energy of a sphere in which $P = K\rho^{1+1/n}$ is given by the expression $E_g = -\frac{3}{5-n}\frac{GM^2}{R}$.

3.2 A brown dwarf is fully convective and may be modeled as an $n = 1$ polytrope. Explain why brown dwarfs for a given K have similar sizes regardless of mass.[2]

3.3 In reality, a "fully" convective star is expected to even possess a thin radiative envelope where the radiative cooling is so efficient that thermal convection subsides. We may construct such a stellar model consisting of an polytropic interior overlaid with a thin radiative layer. Use the condition that all of the thermodynamic quantities (such as T, P and their gradients) need to be continuous across the boundary between the polytropic interior and the radiative outer layer. In addition, it is reasonable to assume that the stellar luminosity L does not vary greatly across the thin radiative outer layer after it emerges from the polytropic interior, because there is no localized source of heating there. Consider an ideal gas. Show that

$$L \propto \frac{k_n^n M^n (T_b R)^{3-n}}{\kappa_b} \left(\frac{\mu}{\mathcal{R}}\right)^n, \tag{5}$$

where k_n is a function of n and the subscript b means the value evaluated at the boundary between the polytropic interior and outer radiative layer.

[2]It can be proved that K is related to specific entropy. Hence, the same K means the same specific entropy, and therefore we compare brown dwarfs with the same "heat" content.