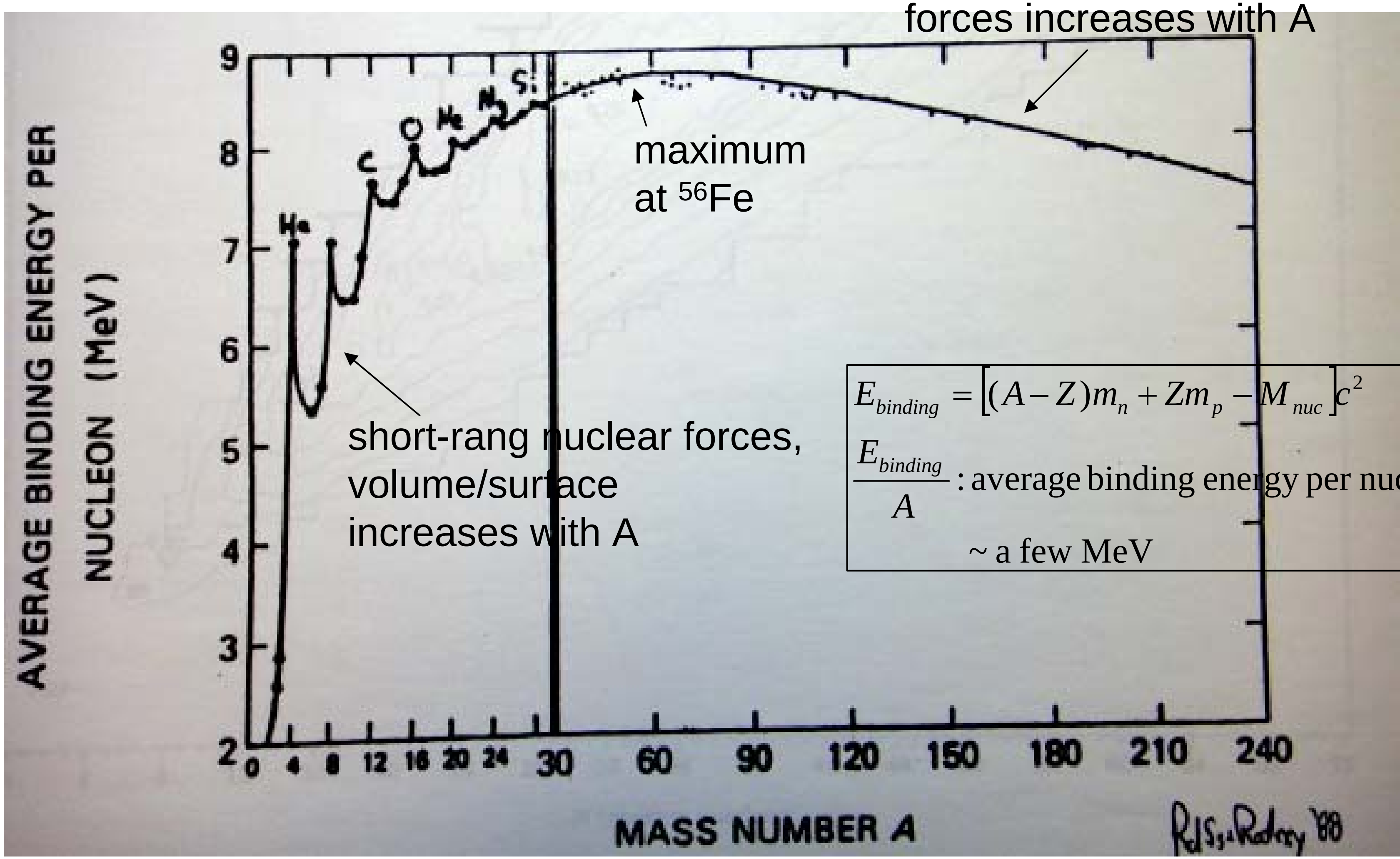
Stellar Physics
Lecture 3: Stellar nucleosynthesis

章品高

gu@asiaa.sinica.edu.tw

nuclear binding energy

repulsion by long-range Coulomb forces increases with A



maximum at ^{56}Fe

short-rang nuclear forces, volume/surface increases with A

$$E_{binding} = \left[(A-Z)m_n + Zm_p - M_{nuc}\right]c^2$$

$\frac{E_{binding}}{A}$: average binding energy per nucleon

~ a few MeV

Quantum tunnelling thru Coulomb barrier

$$E_{Coul} = \frac{Z_1 Z_2 e^2}{r}$$

at the nuclear radius $r_0 \approx A^{1/3} 1.44 \times 10^{-13}$ cm, $E_{Coul} \approx Z_1 Z_2$ Mev.

On the other hand, $E_{th} \approx kT \approx 10^3$ eV at $T \approx 10^7$ K (1eV $\approx 10^4$K), which is $<< E_{Coul}$.

The high-energy tail of the Maxwell-Boltzmann distribution is inadequate!

So, owing to particle-wave duality, quantum tunnelling thru the Coulumb barrier should be included to enhance the reaction.

Formally, we should solve the Schroedinger's equation for two interacting particles in a radial potential:

$$\left[\nabla^2 \psi + \frac{2m_{reduced}}{\hbar^2}(E - V)\psi\right] = 0, \quad \text{with} \quad V(r) = \frac{Z_1 Z_2 e^2}{r} \ \text{for} \ r > r_0 \ \text{and} \ V(r) = V_{nuc} \ \text{for} \ r < r_0.$$

But we may estimate the penetration factor $\exp(-kr)$, please see the next slide.

$$k \approx \frac{1}{\lambda_{\text{de Brogie}}} = \frac{m_{reduced} V_{rel}}{\hbar} = \frac{\sqrt{2m_{reduced} E}}{\hbar} \ \text{and} \ \frac{1}{2} m_{reduced} V_{rel}^2 = \frac{Z_1 Z_2 e^2}{r} \Rightarrow -kr \approx -\frac{2 Z_1 Z_2 e^2}{\hbar V_{rel}}$$

This rough estimate agrees well with the tunnelling probability $= P_{tunnelling} \propto \exp(-\eta)$,

where $\eta = \dfrac{Z_1 Z_2 e^2}{h V_{rel}} = \dfrac{Z_1 Z_2 e^2}{h\sqrt{E / 0.5 m_{reduced}}} \equiv \dfrac{\overline{\eta}}{\sqrt{E}}$.

tunnelling cross sectoin $\sigma(E) \propto P_{tunnelling} \lambda^2_{\text{de Broglie}} = P_{tunnelling} \left(\dfrac{\hbar}{\sqrt{2mE}} \right)^2$

Quantum tunnelling thru Coulomb barrier

- **Schrödinger equation for proton wavefunction:**

$$\frac{d^2\psi}{dr^2} + \frac{2m}{\hbar^2}(E_{\text{kin}} - E_{\text{pot}})\psi = 0.$$

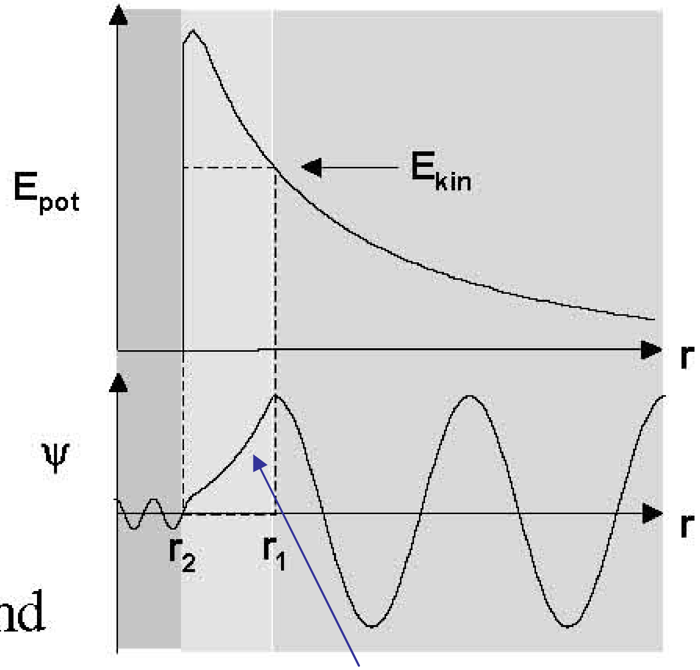
For $r > r_1$ and $r < r_2$,

$(E_{\text{kin}} - E_{\text{pot}}) > 0 \Rightarrow \psi$ is real.

$r > r_1$:	$\psi \sim \sin kr$
$r_2 < r < r_1$:	$\psi \sim e^{-kr}$
$r < r_2$:	$\psi \sim \sigma \sin kr$

where $k = \frac{2m}{h^2}\left(E_{\text{kin}} - E_{\text{pot}}\right)$ and

$\sigma =$ probability of barrier penetration.



"evanescent" wave
Recall spiral density
waves are evanescent
near the corotation resonance
of a disk galaxy

Thermonuclear reaction rate

For n_j particles per unit volume, the total number of reactions per units of volume and time is

$$\tilde{r}_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \sigma V_{rel}$$

But the relative velocity V_{rel} is not uniform but Maxwellian

$$\Rightarrow f(E)dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} \exp(-E/kT) dE,$$

where $E = (1/2) m_{reduced} V_{rel}^2$ and $m_{reduced} = m_j m_k / (m_j + m_k)$.

$$r_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \langle \sigma V_{rel} \rangle,$$

where $\langle \sigma V_{rel} \rangle = \int_0^\infty \sigma(E) V_{rel} f(E) dE \approx \frac{2^{3/2}}{(m_{reduced}\pi)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty \exp\left(-\frac{E}{kT} - \frac{\bar{\eta}}{\sqrt{E}} \right) dE$

$$\propto \int_0^\infty \exp\left(-\frac{E}{kT} - \frac{\bar{\eta}}{\sqrt{E}} \right) dE \equiv \int_0^\infty \exp[J(E)] dE$$

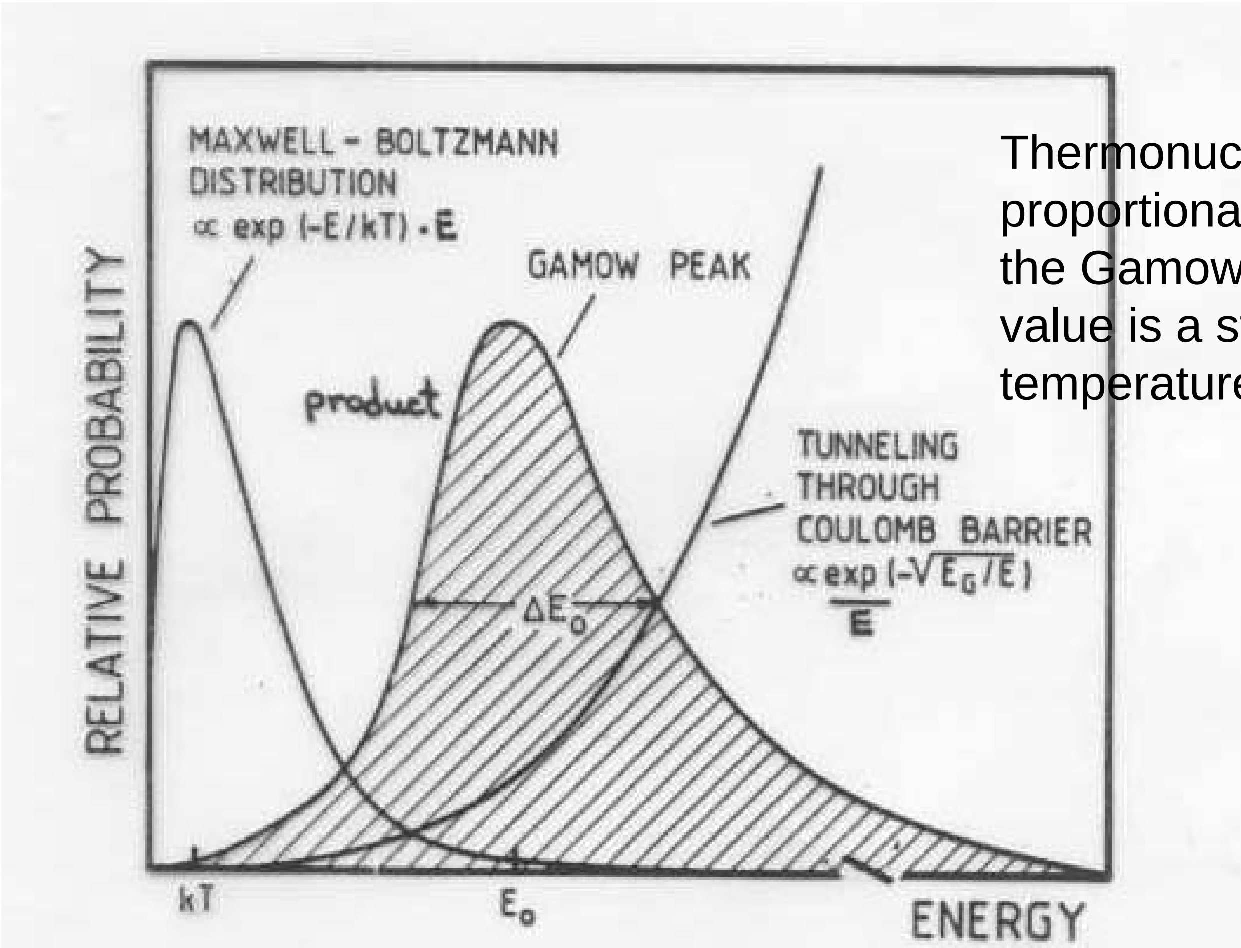
The integrand has appreciable values around its maximum, the so-called Gamow peak.

If the energy Q is released per reaction, the energy generation rate per unit mass:

$$\varepsilon_{\mathrm{jk}} = \rho r_{jk} \frac{1}{1+\delta_{jk}} \frac{Q}{m_j m_k} \rho X_j X_k \langle \sigma V_{rel} \rangle, \quad \text{where } X_i \rho = n_i m_i.$$

The Gamow window

Energy region where there is the best combination of nucleon energies and width of Coulomb barrier that must be penetrated.



Thermonuclear reaction rate proportional to the area below the Gamow peak. The peak value is a steep function of temperature.

Properties of Gamow peak

(1) to first approximation, Gamow peak is Gaussian chracterized by

it mean value E_0, full width at half maximum ΔE_0, and peak value $\exp(-\tau)$:

$$J'(E)=0 \Rightarrow E_0 = \left(\frac{1}{2}\overline{\eta}kT\right)^{2/3}$$

$$J(E)= f_0 + f_0'(E-E_0)+(1/2)f_0''(E-E_0)^2+......$$

$$=-\tau-\frac{1}{4}\tau\left(\frac{E}{E_0}-1\right)^2+....,\quad \text{where}\quad \tau\equiv 3\frac{E_0}{kT}\propto m_{reduced}^{1/3}(Z_iZ_k)^{2/3}(kT)^{-1/3}$$

$$P_{thermonuclear}\propto\int_0^\infty \exp(-J(E))dE=\int_0^\infty \exp\left[-\tau-\frac{\tau}{4}\left(\frac{E}{E_0}-1\right)^2\right]dE\propto kT\tau^{1/2}\exp(-\tau)$$

$$\text{Moreover,}\ \ 0.5\exp(-\tau)=\exp\left[-\tau-\frac{\tau}{4}\left(\frac{E}{E_0}-1\right)^2\right]\Rightarrow\frac{\Delta E_0}{E_0}=4\frac{(\ln 2)^{1/2}}{\tau^{1/2}}$$

(2) The thermonuclear reaction rate strongly sensitive to T:

$$\langle\sigma V_{rel}\rangle\propto\tau^2\exp(-\tau)$$

$$\nu\equiv\frac{\partial\ln\langle\sigma V_{rel}\rangle}{\partial\ln T}=\frac{\tau-2}{3}\quad\text{(i.e. written as a power law)}$$

$T=10^7\,\mathrm{K}\ \Rightarrow\ \tau\approx 20,\ \ \nu\approx 5\Rightarrow$ thermonuclear reaction rate is a steep function of T!

Basic particle interactions

Conservation of leptons and baryons

(1) beta decay:

$$N \rightarrow P + e^- + \overline{\nu}, \qquad P + e^- \rightarrow N + \nu$$

(2) positron decay:

$$P \rightarrow N + e^+ + \nu$$

(3) (P,γ) Process:

$$^{\mathrm{A}}\mathrm{Z} + P \rightarrow {}^{A+1}(\mathrm{Z}+1) + \gamma$$

(4) (α,γ) and (γ,α) Processes

(5) (N,γ) and (γ,N) Processes

Thermonuclear reactions during pre-main sequence phase

temperature increases due to virial theorem

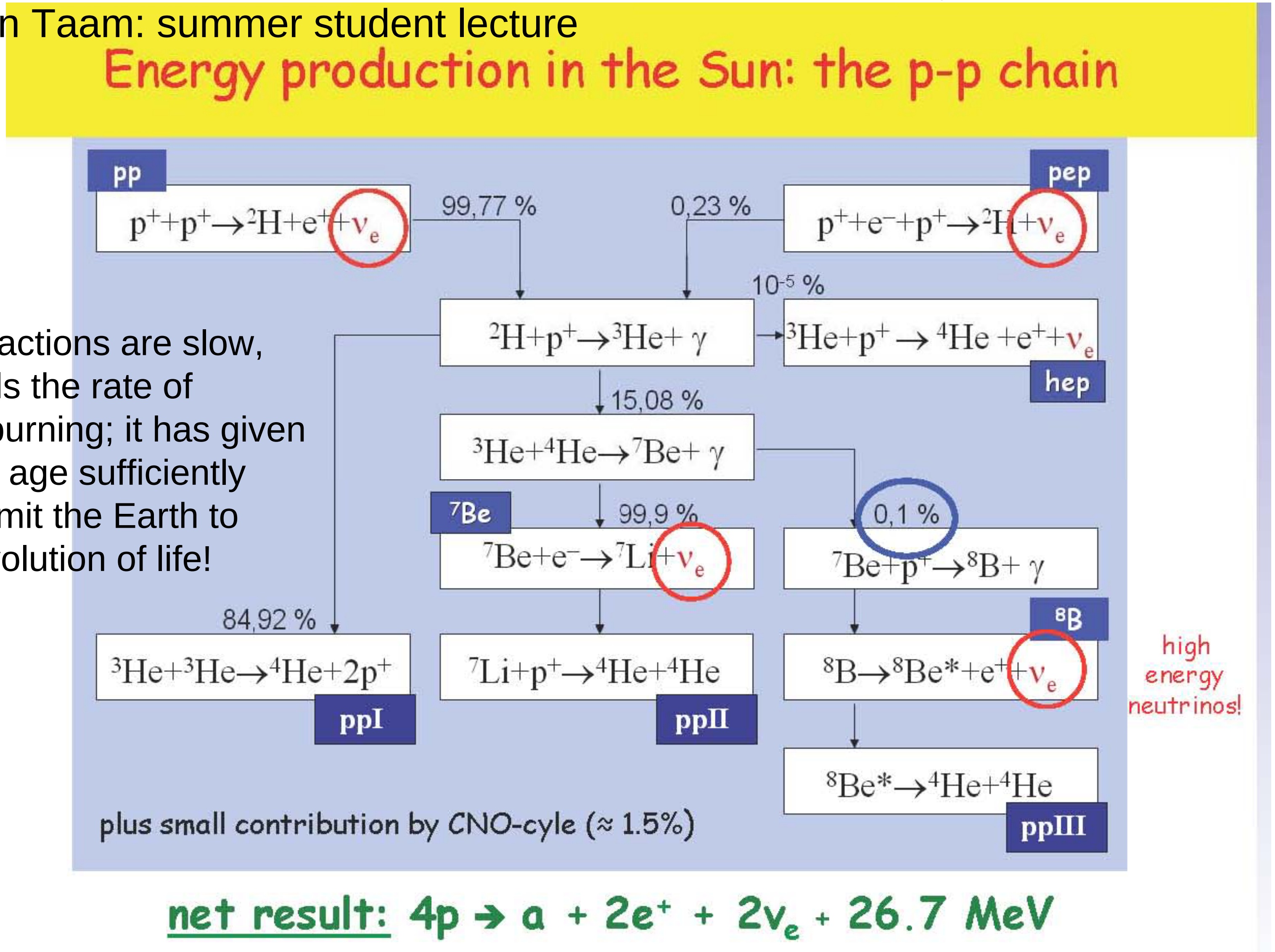
$^{2}D + {}^{1}H \rightarrow {}^{3}He,$	5.5 MeV	~10^{6} K
$^{6}Li + {}^{1}H \rightarrow {}^{3}He + {}^{4}He,$	4.0	~$3\text{x}10^{6}$ K
$^{7}Li + {}^{1}H \rightarrow 2\,{}^{4}He,$	17.3	~$4\text{x}10^{6}$ K
$^{9}Be + 2\,{}^{1}H \rightarrow {}^{3}He + 2\,{}^{4}He,$	6.2	~$5\text{x}10^{6}$ K
$^{10}Be + 2\,{}^{1}H \rightarrow 3\,{}^{4}He + e^{+},$	19.3	~$8\text{x}10^{6}$ K
$^{11}B + {}^{1}H \rightarrow 3\,{}^{4}He,$	8.7	~$8\text{x}10^{6}$ K

- stellar birth line in Hayashi phase
- the theoretical definition of brown dwarfs (M>13 M_Jupiter)
- estimate age by Li abundance (Li is brought by convection to the center of a pre-main sequence star and hence is depleted with time). N.B. ^{7}Li burning also happens in PPII shown in the next slide.

Hydrogen burning ($M \gtrsim 0.08 M_\odot$)

$T \gtrsim 10^7$ K

Ron Taam: summer student lecture

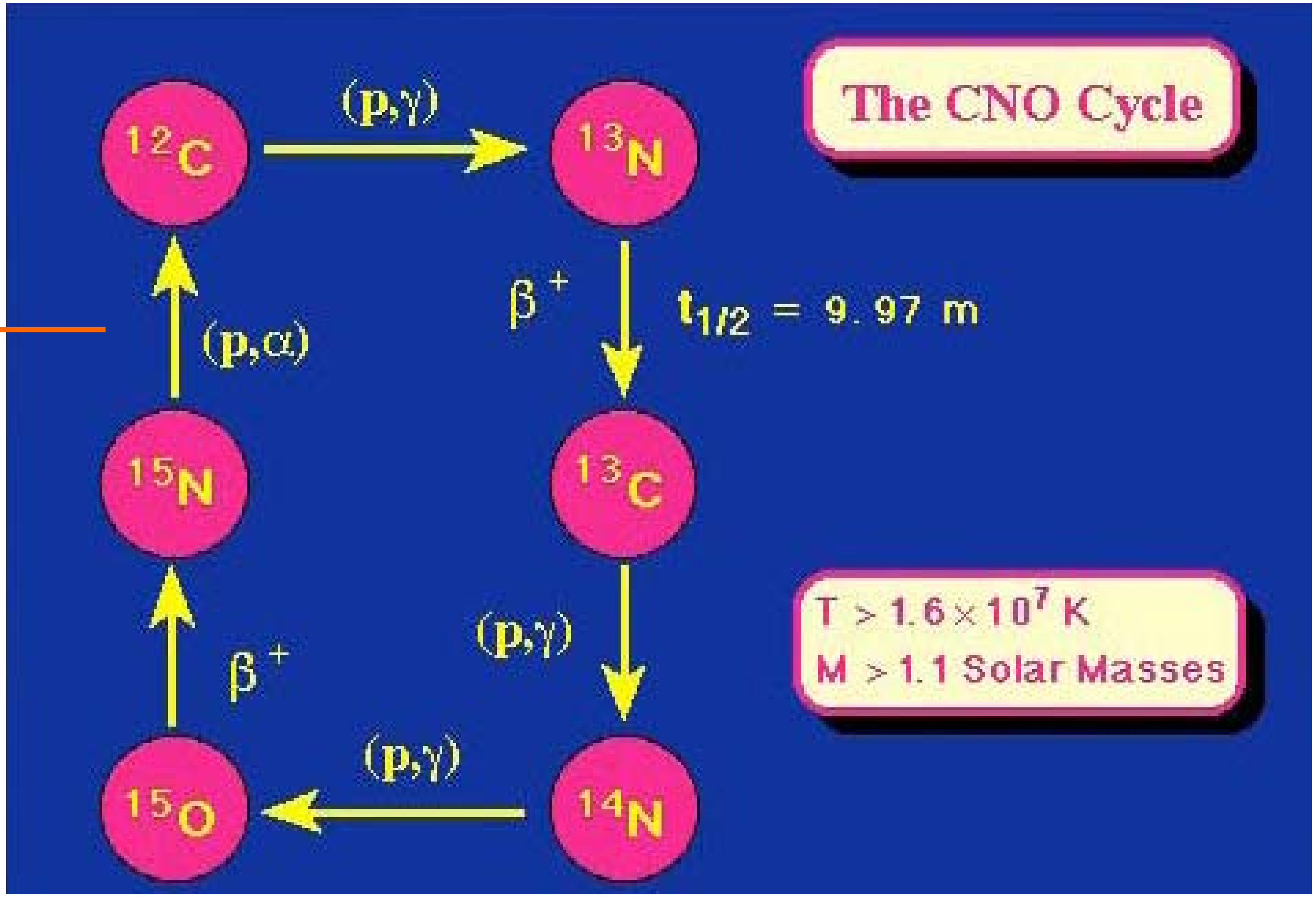


Weak interactions are slow, and controls the rate of hydrogen burning; it has given the Sun an age sufficiently long to permit the Earth to host the evolution of life!

Hydrogen burning

Ron Taam: summer student lecture

^{4}He



CNO cycle vs. PP-chain

Hansen & Kawaler: Stellar Interiors

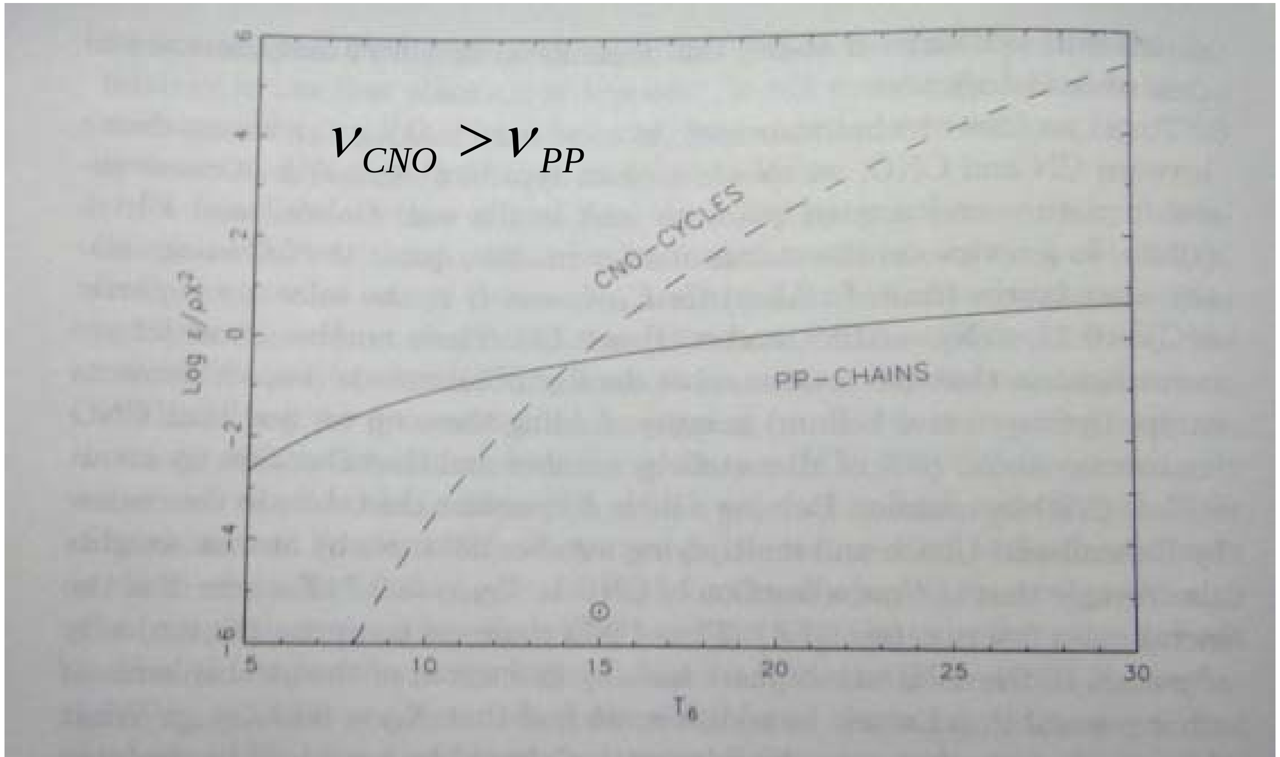
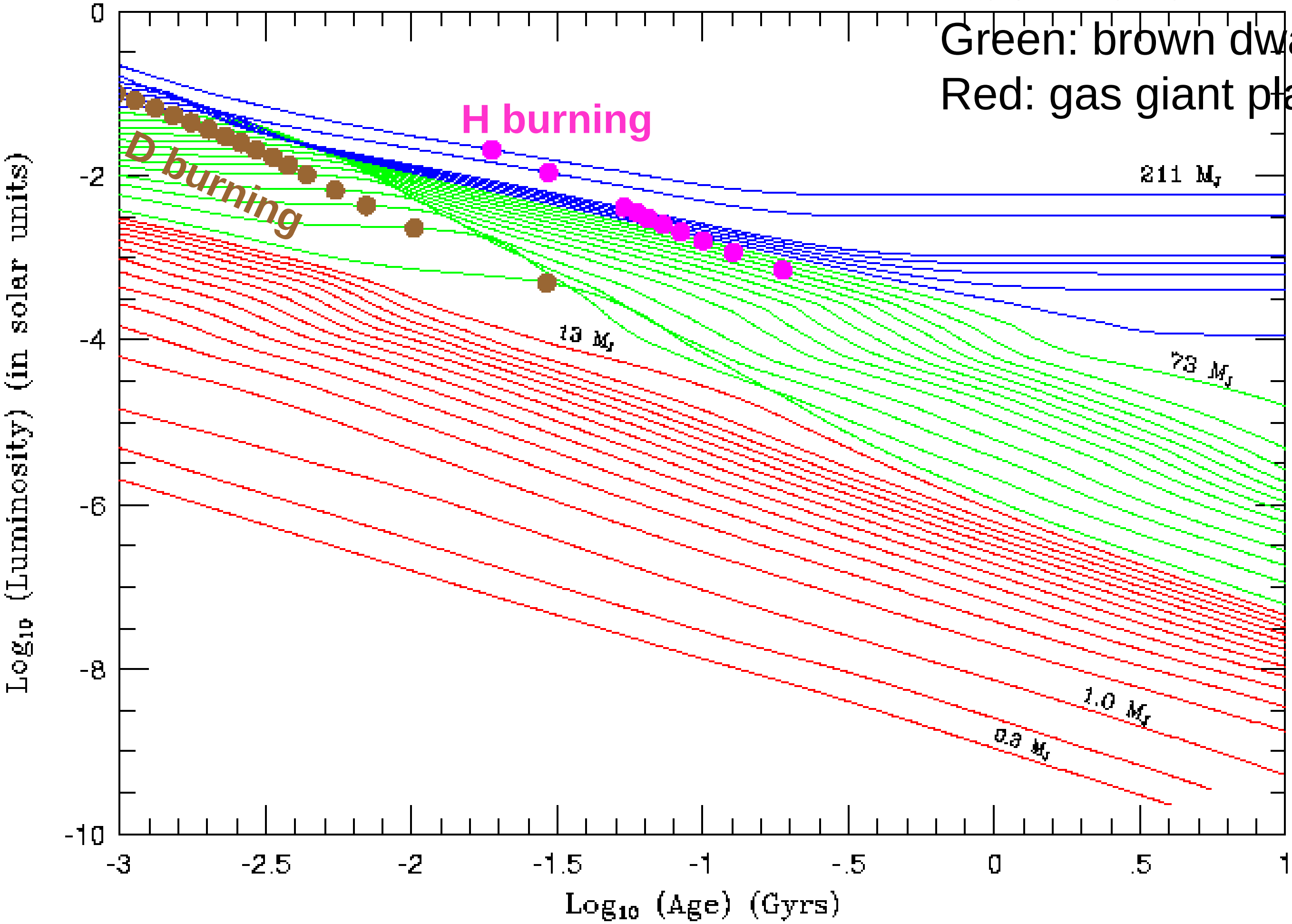


FIGURE 6.7. Plotted as a function of temperature are $\varepsilon_{\rm pp}/\rho X^2$ and $\varepsilon_{\rm CNO}/\rho XZ$. (The legend on the ordinate is generic: "$\varepsilon/\rho X^2$" refers to either depending on context.) To obtain the energy generation rates, you must multiply by the density and the appropriate mass fractions. The temperature of the present-day solar center is indicated by the Sun sign.

Luminosity evolution

Blue: M dwarf
Green: brown dwarf
Red: gas giant planet



http://www.astro.princeton.edu/~burrows/

Helium burning ($M \gtrsim 0.25 M_\odot$)

$T \gtrsim 10^8$ K to overcome the higher Coulomb barrier.

3α reaction:

4He + 4He → ← 8Be

8Be + 4He → 12C + γ

(α, γ) process:

12C + 4He → 16O + γ

16O + 4He → 20Ne + γ

$$Q_{He} \approx 7.28 \times 10^{17} \ \text{erg/g}$$

$$\nu_{3\alpha} > \nu_{CNO} > \nu_{PP}$$

Carbon burning and beyond

12C burning: T$\gtrsim$ 5-10X10^8 K to overcome the even higher Coulomb barrier. Produce nuclei of A=20-24

16O burning: T$\gtrsim$ 10^9 K to overcome the even higher Coulomb barrier. Produce nuclei of A=24-32

When T$\gtrsim$ 10^9 K, photodisintegration becomes important because the radiation contains a significant number of photons with energies in the Mev range, which can break up a nucleus (analogue of photoionization of atoms in atomic processes). For example,

$$\gamma + {}^{28}_{14}\mathrm{Si} \rightarrow {}^{24}_{12}\mathrm{Mg} + \alpha - 9.98\,\mathrm{Mev}$$

The resulting lighter nuclei (e.g. Al, Mg, Ne) will also be subject to photodisintegration, producing more free n, p, and α particles, which can be subsequently captured by Si. When particle capture reactions slightly dominate over photodisintegration, ^{28}Si build up gradually heavier nuclei until ^{56}Fe is reached, Ultimately, two ^{28}Si are converted into ^{56}Fe, which is normally referred to as silicon burning.

Stellar nucleosynthesis beyond Fe

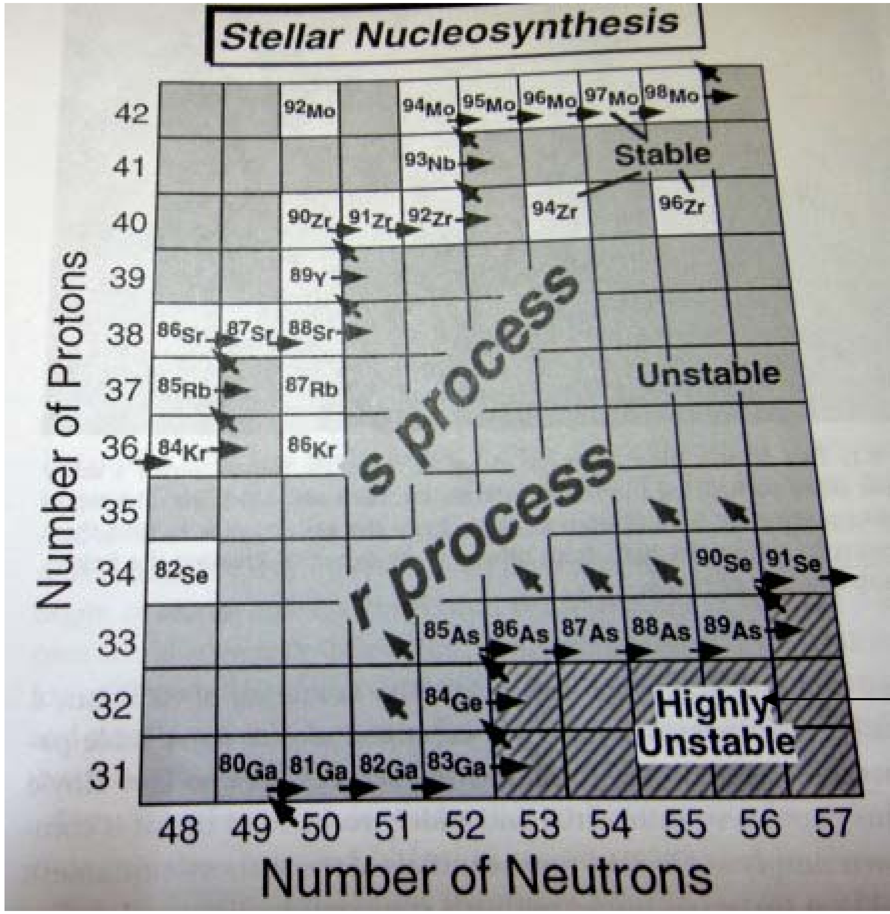
As suggested in the plot for binding energy per nucleon, nuclei synthesized beyond Fe can be done by capture of neutrons to overcome the Coulomb barrier, although these processes are not important to power stars.

$$(Z, A) + n \rightarrow (Z, A+1) + \gamma$$

$$(Z, A+1) \rightarrow (Z+1, A+1) + e^- + \bar{\nu}$$

s-process: s for slow, $\omega_n \ll \omega_\beta$
build up heavier elements close to the line of stability. occurs during the AGB phase.

r-process: r for rapid, $\omega_n \gg \omega_\beta$
occurs during supernova events



McSween: "Meteorites & their parent planets"

due to fast nuclear fissions

Element abundance in the solar system

Williams: Nuclear & Particle Physics

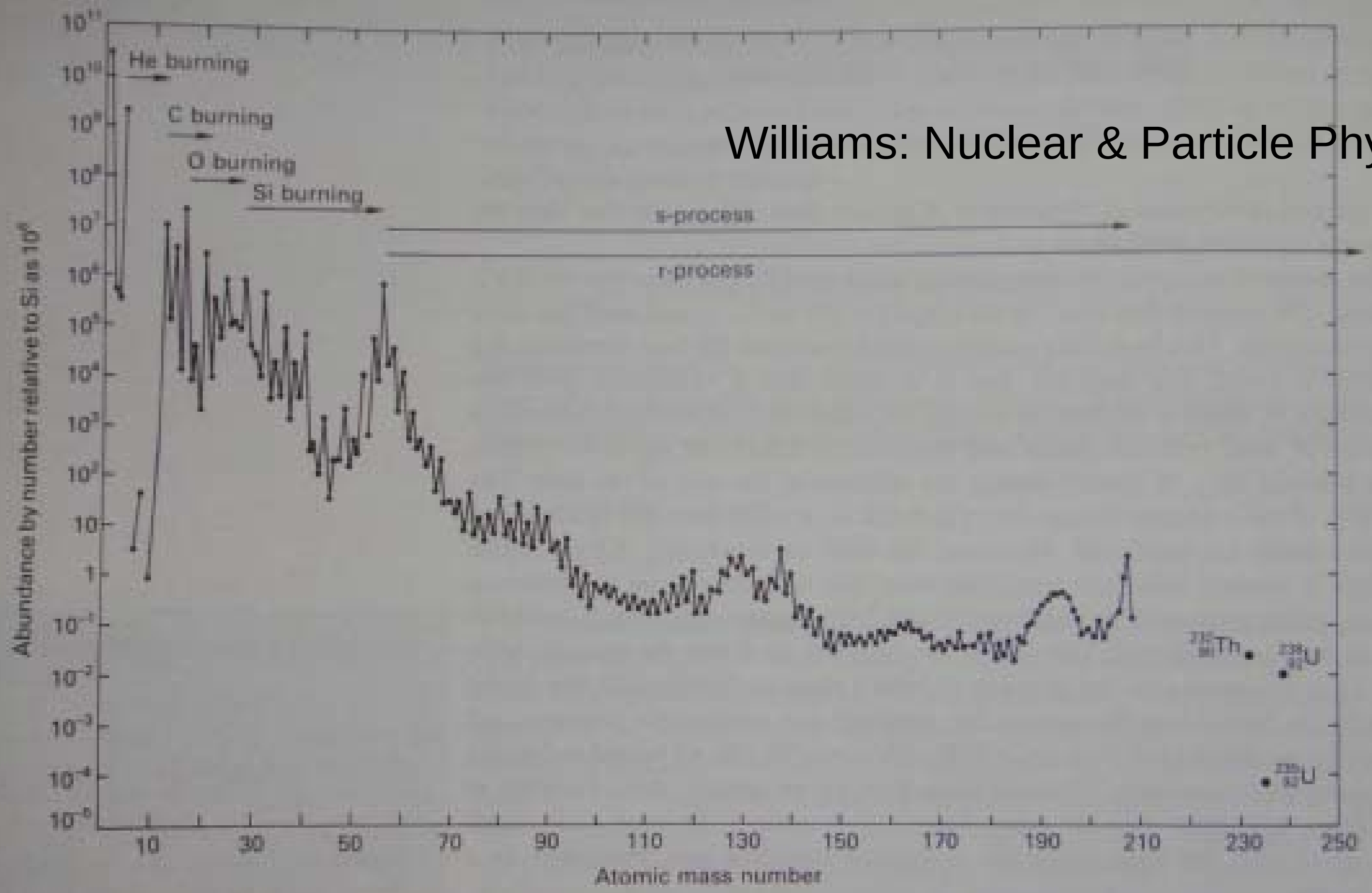


Fig. 14.3 The abundance by number of atoms relative to $^{28}_{14}$Si + $^{29}_{14}$Si + $^{30}_{14}$Si as 10^6 in the solar system, as a function of the atomic mass number A. The relative H, He abundances are close to those existing after Big Bang nucleosynthesis. The obvious absences are those of $A = 5,8$, and $A > 209$ apart from $^{232}_{90}$Th, $^{234}_{92}$U (not shown), $^{235}_{92}$U and $^{238}_{92}$U. Large abundances occur for the even–even nuclei at $A = 12, 16, 20, 24, 28, 32$, and 40. The broad peak near $A = 56$ is the iron-like elements, noted as those with the greatest binding energy per nucleon in the periodic table. The abundance enhancing effect of magic numbers are also seen!

(1) at $A = 86$ to 90 due to $N = 50$,
(2) at $A = 114$ to 120 due to $Z = 50$,
(3) at $A = 138$ due to $N = 82$,
(4) at $A = 208$ due to $Z = 82$, $N = 126$.

In addition, the even-to-odd A abundance due to the pairing term effect on the binding energy is clearly visible.

summary

- fusion & fission: nuclear binding energy release
- Gamow peak: enhance tail of Maxwell distribution thru wave tunneling, determine the temperature sensitivity and hence characterize different thermonuclear syntheses in the course of stellar evolution
- definitions of gas giant planets, brown dwarfs, and stars
- elements heavier than ^{56}Fe: s- and r-processes