Stellar Physics
Lecture 5: stars with polytropic equation of state

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Poisson equation in spherical symmetry

Poisson equation as a result of self-gravity

$$\nabla^2 \phi_{grav} = 4\pi G\rho$$

In hydrostatic equilibrium, $\nabla p = -\rho \nabla \phi_{grav}$

Hence, $\nabla \cdot \left(\frac{1}{\rho} \nabla p \right) = -4\pi G\rho.$

Assume spherical symmetry, the above equation becomes

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dp}{dr} \right) = -4\pi G\rho$$

Polytropes

$p = K\rho^{\Gamma} = K\rho^{(n+1)/n}, \quad \Gamma = \text{polytropic exponent}, \; n = \text{polytropic index}$

e.g. fully convective star $\rightarrow \nabla \approx \nabla_{ad} \rightarrow \Gamma = \gamma =$ adiabatic exponent for an ideal gas

e.g. $\Gamma = 5/3, n = 3/2$ for N.R. ideal monatomic gas

for N.R. degenerate gas

$\Gamma = 4/3, n = 3$ for E.R. ideal monatomic gas

for E.R. degenerate gas

"N.R." = nonrelativistic, "E.R." = extreme relativistic

e.g. $p = p_{gas} + p_{rad} = \dfrac{R\rho T}{\mu} + \dfrac{1}{3} a T^4, \; p_{gas} \equiv \beta p$ and so $p_{rad} = (1-\beta)\, p$

$$\text{solve for } T \Rightarrow T = \left(\frac{R}{\mu} \frac{3}{a} \frac{1-\beta}{\beta} \right)^{1/3} \rho^{1/3} \Rightarrow p = \left[\left(\frac{R}{\mu} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{1/3} \rho^{4/3}$$

If β and μ are not a function of r, then such a gas is an $n = 3$ polytrope.

Lane-Emden Equation

Define $\rho(r) \equiv \rho_c \phi^n(r)$

$\Rightarrow p(r) = K\rho^{(n+1)/n} = K\rho_c^{(n+1)/n}\phi^{n+1} = p_c\phi^{n+1}$

The Poisson equation in spherical symmetry becomes

$$(n+1)K\rho_c^{1/n}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = -4\pi G\rho_c\phi^n$$

define $a \equiv \left[\dfrac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}\right]^{1/2}$ (unit of length) and $\xi \equiv \dfrac{r}{a}$

nondimensionalize the above hydrostatic equation

$$\Rightarrow \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n \quad \text{(Lane - Emden Equation)}$$

2nd order ODE needs 2 boundary conditions :

at $r = \xi = 0$: $\phi(0) = 1$ (identify ρ_c as $\rho(r=0)$. This fixes the normalization of central density),

$$\left.\frac{d\phi}{d\xi}\right|_0 = 0 \quad (dp/dr = 0 \text{ at center by spherical symmetry})$$

For $n < 5$, there is a zero of ϕ at finite ξ, called ξ_1.

$\Rightarrow$ pressure vanishes at ξ_1 $\Rightarrow$ surface of polytrope is reached

$$n = 0: \quad \phi(\xi) = 1 - \frac{\xi^2}{6}, \xi_1 = \sqrt{6}$$

$$n = 1: \quad \phi(\xi) = \frac{\sin\xi}{\xi}, \xi_1 = \pi$$

$$n = 5: \quad \phi(\xi) = \left(1 + \frac{\xi^2}{3}\right)^{-1/2}, \xi \to \infty$$

Table 2-5
Constants of the Lane-Emden functions†

n	ξ_1	$-\xi_1^2 \left(\frac{d\phi}{d\xi}\right)_{\xi=\xi_1}$	$\frac{\rho_c}{\bar{\rho}}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6,189.47
4.9	169.47	1.7355	934,800
5.0	∞	1.73205	∞

† S. Chandrasekhar, "An Introduction to the Study of Stellar Structure," p. 96; reprinted from the Dover Publications edition, Copyright 1939 by The University of chicago, as reprinted by permission of The University of Chicago.

Explicit solutions exist only for n=0,1, and 5. Otherwise, a numerical solution is necessary.

Properties of polytrope

radius: $R_* = a\xi_1 = \left[\frac{(n+1)K}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n}\xi_1$

mass: $m(\xi) = \int_0^{a\xi} 4\pi r^2 \rho dr = 4\pi a^3 \int_0^{\xi} \rho_c \phi^n \xi^2 d\xi$

$$= -4\pi a^3 \rho_c \int_0^{\xi} \frac{d}{d\xi}\left(\xi^2 \frac{d\phi}{d\xi}\right) d\xi = -4\pi a^3 \rho_c \xi^2 \frac{d\phi}{d\xi}$$

$$M_* = m(\xi = \xi_1) = -4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left(\xi^2 \frac{d\phi}{d\xi}\right)_{\xi_1}$$

$M_* - R_*$ relation: $K = \left[\frac{4\pi}{\xi^{n+1}(-\phi')^{n-1}}\right]_{\xi_1}^{1/n} \frac{G}{n+1} M_*^{1-1/n} R_*^{-1+3/n}$

central pressure: $p_c = K\rho_c^{1+1/n} = \frac{1}{4\pi(n+1)(\phi')_{\xi_1}^2} \frac{GM_*^2}{R_*^4}$

central concentration: $\bar{\rho} = \frac{M_*}{\frac{4}{3}\pi a^3 \xi_1^3} \Rightarrow \frac{\bar{\rho}}{\rho_c} = -\frac{3}{\xi_1} \left.\frac{d\phi}{d\xi}\right|_{\xi_1}$

Given M_* and K, one can solve the interior structure for a particular polytropic index n.

Recall that for an ideal gas

$$s \propto \ln\left(\frac{p^{1/\gamma}}{\rho}\right)$$

when we studied thermal convection. It can be shown that

$$s \propto \ln\left(\frac{p^{1/\Gamma}}{\rho}\right) \propto \ln K$$

for a polytropic equation of state. So, K represents the stellar entropy.

Distribution function

Statistical mechanics:

$$n(p) = \frac{1}{h^3} \sum_j \frac{g_i}{\exp\left[(\varepsilon(p) - \varepsilon_F)/kT\right] \pm 1}$$

degeneracy of state j (statistical weight)

Number density
In phase space

energy of state j

Kinetic energy
as a function of
momentum p

Fermi
energy

+: fermions
-: bosons
ε - ε_F>>kT:
classic fermions
(Boltzmann distribution)

$n = \int n(p) 4\pi p^2 dp$

kinetic energy : $\varepsilon(p) = (p^2c^2 + m^2c^4)^{1/2} - mc^2$

Hamilton's equation : $v = \frac{\partial \varepsilon}{\partial p}$

$P = \frac{1}{3} \int n(p) p v 4\pi p^2 dp$

$E = \int n(p) \varepsilon(p) 4\pi p^2 dp$

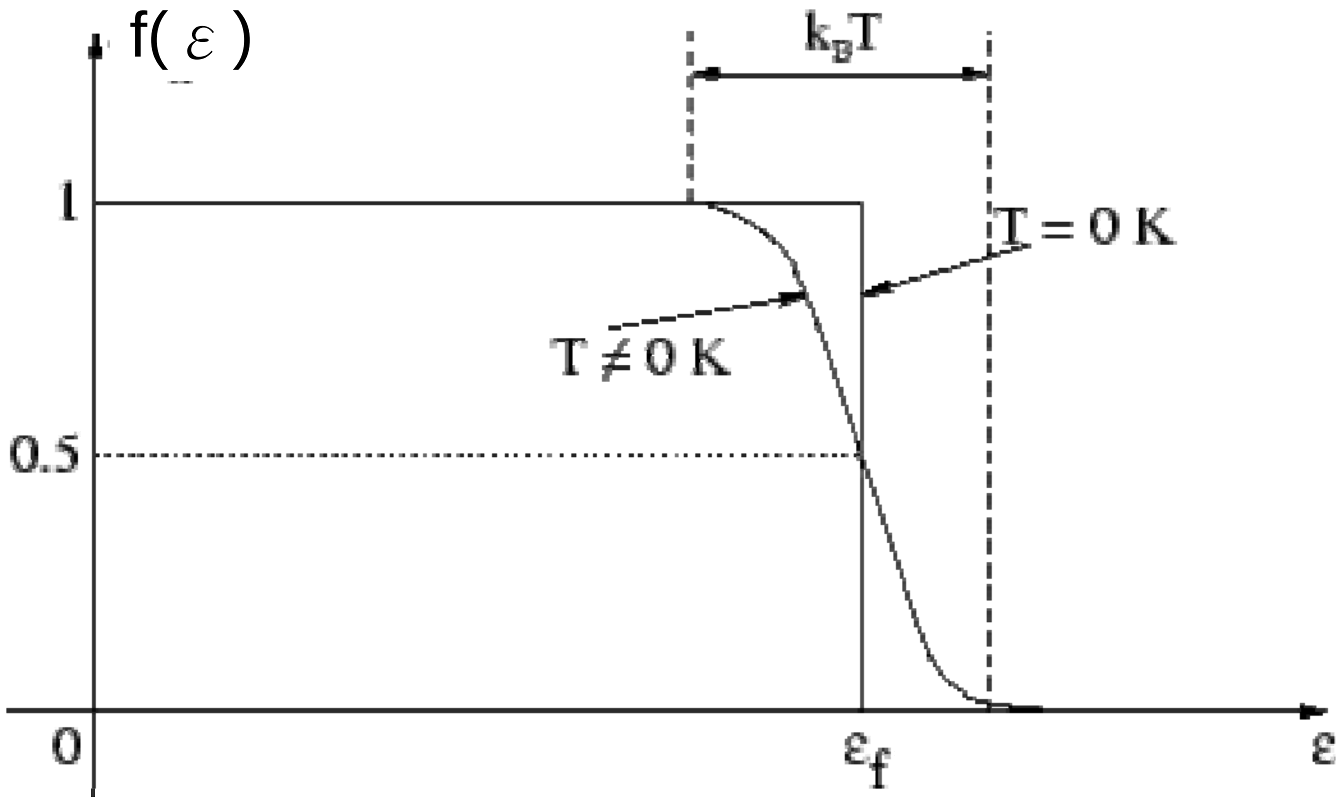
for example: photon gas (bosons)
g=2 (2 polarizations), ε =*pc*, ε_F=0

Fermi-Dirac distribution

Probability that
A particle will
Have energy ε

$$f(\varepsilon) = \frac{1}{\exp\left(\frac{\varepsilon - \varepsilon_F}{kT}\right) + 1}$$

degree of degeneracy
depends on ε_F/kT



application 1: complete degenerate gas

$$n = \frac{2}{h^3}\int_0^{\varepsilon_F} f(\varepsilon)4\pi p^2 dp = \frac{8\pi}{h^3}\int_0^{\varepsilon_F} p^2 dp = \frac{8\pi}{3}\left(\frac{h}{mc}\right)^{-3} x_F^3,$$

where $x_F \equiv p_F / mc$ and note that $\dfrac{h}{m_e c} = 0.024\,\mathring{\mathrm{A}}$ is the Compton wavelength for e^-.

For a completely degenerate electron gas, $n_e = \dfrac{8\pi}{3}\left(\dfrac{h}{m_e c}\right)^{-3} x_F^3.$

Using $n_e = \dfrac{\rho}{\mu_e m_u}$, the above equation becomes $\dfrac{\rho}{\mu_e} = \dfrac{8\pi m_u}{3}\left(\dfrac{h}{m_e c}\right)^{-3} x_F^3$

For isotropic pressure $P_e = \dfrac{1}{3}\displaystyle\int n(p) pv 4\pi p^2 dp = \left[\dfrac{\pi}{3}\left(\dfrac{h}{m_e c}\right)^{-3}\right] m_e c^2 F(x_F),$

where $F(x_F) = (8/5)x_F^5 - (4/7)x_F^7 + \ldots.$ for $x_F << 1$ (N.R. particles)

$F(x_F) = 2x_F^4 - 2x_F^2 + \ldots.$ for $x_F >> 1$ (E.R. particles)

Hence,

$$P_e = \frac{3^{2/3}\pi^{4/3}}{5}\frac{\hbar^2}{m_e m_u^{5/3}}\left(\frac{\rho}{\mu_e}\right)^{5/3} = 1.004\times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \text{ dyne cm}^{-2} \text{ for N.R.}$$

$$P_e = \frac{3^{1/3}\pi^{2/3}}{4}\frac{\hbar c}{m_u^{4/3}}\left(\frac{\rho}{\mu_e}\right)^{4/3} = 1.243\times 10^{15}\left(\frac{\rho}{\mu_e}\right)^{4/3} \text{ dyne cm}^{-2} \text{ for E.R.}$$

White dwarf

For low-density W.D., the elections are N.R., so Γ =5/3 is a good description. As the density increases, the electrons become relativistic, first at the center and eventually throughout the star. In the high-density case, Γ =4/3 is a good description. We can derive a mass-radius relation for W.D. in these limits.

For $\Gamma = 5/3, n = 3/2 \Rightarrow \xi_1 = 3.65375, \left| \xi^2 \frac{d\phi}{d\xi} \right|_{\xi_1} = 2.714$

$M_* - R_*$ relation $\Rightarrow M_* = 0.7011 \left(\frac{R_*}{10^4 \text{ km}} \right)^{-3} \left(\frac{\mu_e}{2} \right)^{-5} M_{sun},$

where $R_* = 1.122 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g km}^{-3}} \right)^{-1/6} \left(\frac{\mu_e}{2} \right)^{-5/6}$ km.

For $\Gamma = 4/3, n = 3 \Rightarrow \xi_1 = 6.89695, \left| \xi^2 \frac{d\phi}{d\xi} \right|_{\xi_1} = 2.018$

$\Rightarrow M_* = 1.457 \left(\frac{2}{\mu_e} \right)^2 M_{sun}$, independent of ρ_c and R_*!

$R_* = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g/cm}^3} \right)^{-1/3} \left(\frac{\mu_e}{2} \right)^{-2/3}$ km.

There is an important result contained in these equations: ρ_c increases as M_* increases in the low-density regime. Hence, as M_* increases, the white dwarf becomes more relativistic. Eventually, as $\rho_c \rightarrow \infty$ and $R_* \rightarrow 0$, M_* approaches the value in the high density limit, a.k.a the Chandrasekhar limit.

application 2: Hayashi line

For simplicity, assume a fully ionized ideal gas in a fully convective star.

$$\nabla \approx \nabla_{\mathrm{ad}} \Rightarrow P = C_n T^{1+n}, \text{ where } n = \frac{1}{\nabla_{ad}} - 1 = \frac{3}{2}.$$

$$\text{Apply } P = \frac{R\rho T}{\mu} \text{ and } P = K\rho^{(n+1)/n} \Rightarrow C_n = K^{-n}\left(\frac{R}{\mu}\right)^{1+n}$$

Use the mass-radius relation: $K \propto M_*^{1/3} R_*$

$\Rightarrow C_{n=3/2} = C'_{n=3/2} R_*^{-3/2} M_*^{-1/2}$, where C'_n is known for given n and μ.

Finally, stellar interior is described by

$$\log T = 0.4 \log P + 0.4\left(\frac{3}{2}\log R_* + \frac{1}{2}\log M_* - \log C'\right)$$

application 2: Hayashi line

For a more realistic model, consider the photospheric condition to replace the zero outer B.C. in the polytrope:

at the photosphere: $P_p = \dfrac{2g_p}{3\kappa_p} \propto \left(\dfrac{M_*}{R_*^2 T_{eff}^b}\right)^{\frac{1}{a+1}}$ with $\kappa \propto P^a T^b$

$\Rightarrow (a+1)\log P_p = \log M_* - 2\log R_* - b\log T_{eff} + \text{const.}$

Combining the polytropic interior and the photophspheric boundary, we obtain the Hayashi line in the Hertzsprung - Russell (HR) diagram:

$\log T_{eff} = A\log L_* + B\log M_* + \text{const.}$

where $L_* = 4\pi R_*^2 \sigma T_{eff}^4$, $\quad A = \dfrac{0.75a - 0.25}{b + 5.5a + 1.5}$, $\quad B = \dfrac{0.5a + 1.5}{b + 5.5a + 1.5}$

For H^- dominant opacity, $a \approx 0.5$, $\mathrm{b} \approx 8.5 \Rightarrow A \approx 0.01$, $B \approx 0.14$

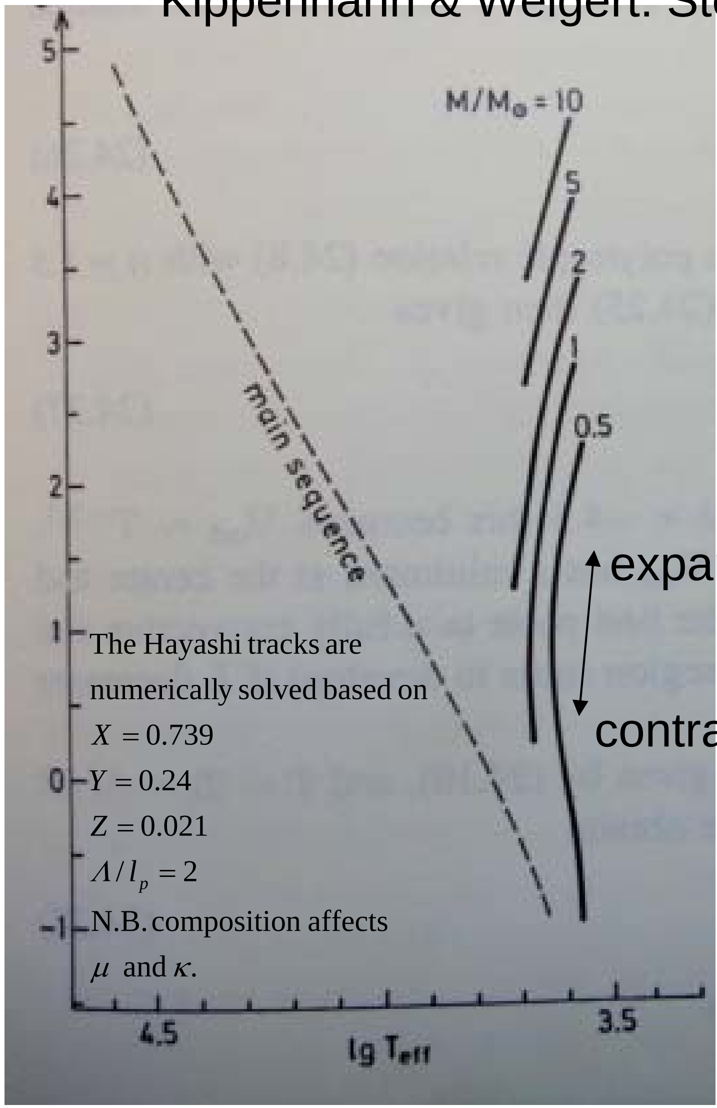
$\Rightarrow \dfrac{\partial \log L_*}{\partial \log T_{eff}} = \dfrac{1}{A} >> 1 \Rightarrow$ Hayashi line is very steep!

Some books use modest opacity power law: $a \approx 1$, $\mathrm{b} \approx 3$, but still get $\dfrac{1}{A} >> 1$.

application 2: Hayashi line

Kippenhahn & Weigert: Stellar Structure and Evolution

log $L/L_\odot$



The Hayashi tracks are numerically solved based on

$X = 0.739$

$Y = 0.24$

$Z = 0.021$

$\Lambda / l_p = 2$

N.B. composition affects μ and κ.

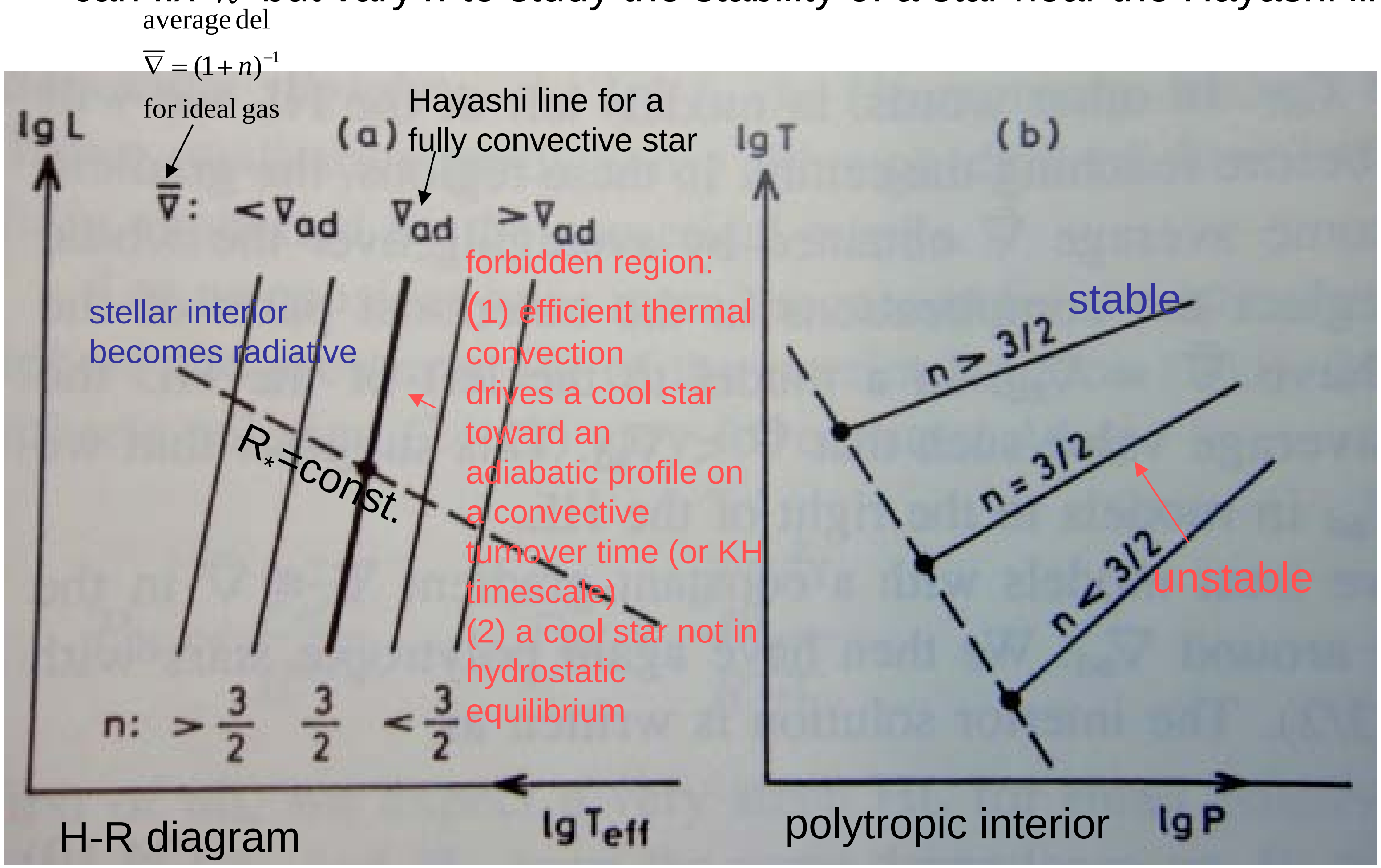
$$L = 4\pi R_*^2 \sigma T_{eff}^4$$

expansion (RGB,AGB)

contraction (pre-main sequence)

application 2: Hayashi line

In the previous calculation, we fix *n* (=3/2) to find Hayashi lines. Now we can fix κ but vary *n* to study the stability of a star near the Hayashi line.



Kippenhahn & Weigert: “Stellar Structure & Evolution”