Stellar Physics: Midterm Exam - Apr 25, 2014

Name:
Student ID:

Please work all of the following 5 problems. No notes, books, internet resources to be used. Some constants (in cgs units) that you may find useful are: $G = 6.67 \times 10^{-8}$, $h = 6.626 \times 10^{-27}$, $e = 4.8 \times 10^{-10}$, $m_u = 1.66 \times 10^{-24}$, $k = 1.38 \times 10^{-16}$, eV$= 1.6 \times 10^{-12}$, $\sigma = 5.67 \times 10^{-5}$, $M_\odot = 1.99 \times 10^{33}$, $R_\odot = 6.96 \times 10^{10}$, $L_\odot = 3.9 \times 10^{33}$.

1. (20 pts) In the first homework set, you were asked to solve the interior structure of a star of total mass M and radius R for a constant-density model (i.e. $\rho =$ constant) and for a linear-density model (i.e. $\rho(r) \propto 1 - r/R$), subject to the zero outer boundary condition, i.e. $P(R) = T(R) = 0$. Assume that the star is composed of an ideal monatomic gas. Determine the range in radius for the convectively unstable region in these two stellar models.
Solution: Thermal convection occurs in an ideal monatomic gas when $(d\ln\rho/d\ln P)_{ambient} < (d\ln\rho/d\ln P)_{blob} = 1/\gamma = 3/5$, where the density change of the blob has been assumed adiabatic. In a constant-density model,
$$(d\ln\rho/d\ln P)_{ambient} = 0 < 3/5 \Longrightarrow \text{ the entire star is convective.} \tag{1}$$
In a linear-density model, let $x = r/R$ and then $\rho(x) = \rho_c(1-x)$. Furthermore, $dP/dr = -\rho GM(r)/r^2$, $M_r = \int_0^r 4\pi r^2 \rho_c(1 - r/R)dr$, and $P(x = 1) = 0$ give $P(x) = P_c(1 - 24x^2/5 + 28x^3/5 - 9x^4/5) = P_c[1 + 2x - (9/5)x^2](1-x)^2$. Hence, thermal convection occurs when
$$\begin{aligned}(d\ln\rho/d\ln P)_{ambient} &= \frac{d\ln(1-x)}{dx} \cdot \frac{dx}{d\ln[(1+2x-\frac{9}{5}x^2)(1-x)^2]} \qquad (2)\\ &= \frac{5+10x-9x^2}{12x(4-3x)} < \frac{3}{5}\\ &\Longrightarrow 0.346 < x < 1.145\\ &\Longrightarrow \text{ convection zone lies in the region } r \geq 0.346R.\end{aligned}$$

2. (20 pts)

   2.1 Use the de Brogie's relation for the particle-wave duality, the Coulomb potential barrier, and the Maxwell-Boltzmann velocity distribution. Derive the thermonuclear reaction rate $\propto \int_0^\infty \exp(-E/kT - \bar{\eta}/\sqrt{E})dE$, i.e. the area below the so-called "Gamow peak". In the equation, E is the relative kinetic energy between

two nuclei, T is the temperature, and $\bar{\eta} \approx Z_1 Z_2 e^2 \sqrt{(m_u/2)[A_1 A_2/(A_1 + A_2)]}/h$ with Z being the atomic number and A being the mass number of each nucleus.
Solution: refer to the class notes.

2.2 Find the value of the kinetic energy $E = E_0$ that gives rise to the maximum value of the Gamow peak. Consider the pp chain reaction at $T = 10^7 K$. Calculate the energy ratio E_0/kT. What does the ratio physically mean?
Solution: The maximum value of the Gamow peak is obtained from $d(-E/kT - \bar{\eta}/\sqrt{E})/dE = 0 \Longrightarrow E_0 = [(1/2)\bar{\eta}kT]^{2/3}$. Hence,

$$\frac{E_0}{kT} = \frac{[(1/2)\bar{\eta}kT]^{2/3}}{kT} \approx 0.45. \tag{3}$$

The ratio $\lesssim 1$ implies that the quantum tunneling does not enhance the pp-chain reaction rate for thermal protons in the high energy tail of the Maxwellian energy distribution against the Coulomb barrier. As a result, the reaction rate is less sensitive to temperature than it ought to be.
Comment: in reality, E_0/kT should be larger than 1 as we have learned in class. The error comes from the simplified expression for $\bar{\eta}$ we use, which underestimates $\bar{\eta}$. The more accurate expression is given by $\bar{\eta} = \pi(2m_{reduced})^{1/2} Z_1 Z_2 e^2/\hbar$, leading to $E_0/kT \approx (4\pi^2)^{2/3}(0.45) = 5.2 > 1$.

3. (20 pts) Consider a pre-main-sequence star of one solar mass on the Hayashi track. Its luminosity is $10L_\odot$ and its effective temperature is $T_{eff} = 4000$ K.

3.1 Ignore any thermonuclear heating and any mass accretion onto the star. Estimate the contraction timescale of the star due to radiative cooling from the stellar surface.
Solution:
The radius of the star is given by $R_* = \sqrt{L_*/4\pi\sigma T_{eff}^4} = \sqrt{10L_\odot/4\pi\sigma(4000K)^4} \approx 4.62 \times 10^{11}$ cm. The contraction time can then be estimated:

$$t_{KH} = \frac{GM_*^2}{R_* L_*} \approx \frac{GM_\odot^2}{(4.62 \times 10^{11}\,\mathrm{cm})(10L_\odot)} \approx 0.46\,\mathrm{Myrs}. \tag{4}$$

3.2 Calculate the mean molecular weight and the number density n at the photosphere of the star. Take the Rosseland mean opacity $\kappa_R = 0.1$ cm^2/g, $X = 0.74$, $Y = 0.25$. Assume that the gases are ideal, monatomic, and all neutrals. Ignore the contributions from elements heavier than He.
Solution: The mean molecular weight for an ideal, monatomic, electrically neutral gas consisting of H and He is given by

$$\mu = \left(\frac{X}{\mu_H} + \frac{Y}{\mu_{He}}\right)^{-1} = \left(\frac{0.74}{1} + \frac{0.25}{4}\right)^{-1} \approx 1.246. \tag{5}$$

The gas pressure at the photosphere is given by $P = (2/3)g/\kappa = (2/3)GM_*/R_*^2\kappa \approx 4.15 \times 10^3$ dyne/cm^2. The number density is obtained from the ideal gas law:

$$n = \frac{P}{kT_{eff}} \approx 7.52 \times 10^{15}\,\mathrm{cm}^{-3}. \tag{6}$$

3.3 Calculate the thermal ionization fraction $\equiv n_e/n$ contributed from elements K, Na, and H at the photosphere. Assume that the ions are far less than the neutrals for each elements. The ionization potentials of K and Na are $\chi_K = 4.34$ eV and χ_{Na} =5.14 eV. Take their abundances by number $n_K/n = 10^{-7}$ and $n_{Na}/n = 10^{-5.5}$.

Solution: For a weakly ionized gas, $n_j^+ << n_j$ and the Saha equation is reduced to $n_j^+/n_j \approx (m_e kT/2\pi\hbar^2)^{3/2}\exp(-\chi_j/kT)$. Summing over all species, we can obtain the thermal ionization fraction $n_e/n \approx [\sum_j (f_j/n)(m_e kT/2\pi\hbar^2)^{3/2}\exp(-\chi_j/kT)]^{1/2} \equiv [\sum_j (n_e/n)_j^2]^{1/2}$, where $f_j \equiv n_j/n$. Given that $f_K = 10^{-7}$, $f_{Na} = 10^{-5.5}$, and $f_H = (\rho_H/\mu_H)/(\rho/\mu) = (\mu/\mu_H)X \approx 0.922$, the thermal ionization fraction is given by

$$\begin{aligned} \frac{n_e}{n} &\approx \left[\left(\frac{n_e}{n}\right)_K^2 + \left(\frac{n_e}{n}\right)_{Na}^2 + \left(\frac{n_e}{n}\right)_H^2\right]^{1/2} \\ &\approx (2.76 \times 10^{-8} + 8.59 \times 10^{-8} + 5.61 \times 10^{-13})^{1/2} \approx 3 \times 10^{-4}. \end{aligned} \tag{7}$$

Comment: the results show that the assumption for weak ionization of the alkaline elements K and Na is extremely poor. One should numerically solve the full version of the Saha equation for the thermal ionization fraction.

3.4 Assume that the interior can be modeled as an $n = 3/2$ polytrope. Calculate how much energy needs to be radiated away for the star to contract along the Hayashi track down to $L = L_\odot$ and $T_{eff} = 4500$ K.

Solution: The stellar radius at 4500 K is given by $R_* = \sqrt{L_*/4\pi\sigma T_{eff}^4} = \sqrt{L_\odot/4\pi\sigma(4500K)^4} \approx 1.16 \times 10^{11}$ cm. The energy radiated away is given by the difference of the total energy:

$$W_f - W_i = -\alpha_{n=3/2}\frac{GM_*^2}{2r_f} + \alpha_{n=3/2}\frac{GM_*^2}{2r_i} \approx -8.5 \times 10^{47}(\alpha_{n=3/2})\ \mathrm{erg}. \tag{8}$$

where $\alpha_{n=3/2}$ needs to be solved numerically using the Lane-Emden Equation but it should be on the order of unity.

4. (20 pts) The concept of the Schönberg-Chandrasekhar limit for a collapse of an isothermal inert He core in a star can apply to the condition for star formation in an interstellar cloud.

   4.1 Derive the virial theorem for a self-gravitating interstellar cloud of total mass M and radius R confined by an external pressure of P_0. Assuming that the cloud has a constant temperature T.
   Solution: Refer to the slide for the "Schönberg-Chandrasekhar limit. Starting from the hydrostatic equilibrium, we have

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \Longrightarrow \int_{cloud} 4\pi r^3 dP = -\int_{cloud} \frac{Gm}{r} dm = E_g, \tag{9}$$

   where E_g is the gravitational energy of the cloud. Integration by parts on the LHS gives

$$4\pi R^3 P_0 - 3\int_{cloud} 4\pi P r^2 dr = 4\pi R^3 P_0 - 2\int_{cloud} (3/2) nkT 4\pi r^2 dr = 4\pi R^3 P_0 - 2E_i, \tag{10}$$

   where E_i is the internal energy ($E_i = (3/2)(RT/\mu)M$ for a constant T). Hence, the virial theorem for the cloud confined by an external pressure is given by

$$4\pi R^3 P_0 = 2E_i + E_g. \tag{11}$$

   4.2 Holding M and T fixed, show that a maximum value, P_{max}, exists for the external pressure for a certain value of the cloud radius, R_{max}. Derive the expressions for R_{max} and P_{max} in terms of the cloud mass M, temperature T, and fundamental constants such as G. In what condition does the cloud gravitationally collapse?
   Solution: Following up the above result (or referring to the class notes), we have

$$P_0 = \left(\frac{3RT}{4\pi\mu}\right)\frac{M}{R^3} - \left(\frac{\alpha G}{4\pi}\right)\frac{M^2}{R^4}, \tag{12}$$

   where $\alpha \equiv -E_g/(GM^2/R)$ and is an order of unit constant. At R_{max}, $dP_0/dR = 0$, which yields

$$\left(\frac{-9RT}{4\pi\mu}\frac{M}{R^4} + \frac{4\alpha G}{4\pi}\frac{M^2}{R^5}\right)_{R=R_{max}} = 0 \Longrightarrow R_{max} = \left(\frac{4\alpha G\mu}{9RT}\right)M. \tag{13}$$

   Thus,

$$P_{max} = \left(\frac{3^7 R^4}{4^5\pi\alpha^3 G^3\mu^4}\right)\frac{T^4}{M^2}. \tag{14}$$

   The cloud collapses when $P_0 > P_{max}$.

5. (20 pts) Sketch and explain the evolution of a $1 M_\odot$ star from a pre-main-sequence star to a cool white dwarf on the H-R diagram. You may draw the pre-main sequence evolution and the main sequence and post-main sequence evolution on separate H-R diagrams to avoid crowding.

   Solution: refer to the class notes. There are a number of evolutionary states, but it should be noted that the gravitational core collapse are as important as thermonuclear reactions throughout the stellar evolution: stellar birth line, Hayashi track, Henyey track, ZAMS (pp-chain H burning begins), MS (luminosity gradually increases), Schönberg-Chandrasekhar limit, subgiant (He core contracts, H shell burning), RGB (He core keeps contracting and becomes degenerate), He core flash, horizontal branch (He core burning, H shell burning), AGB (H and He shell burning, C/O contracts and becomes degenerate), AGB winds (He shell flashes), PNe, WD