Galactic Dynamics: Homework I

due in class on Mar 10th, 2008

1. **Relaxation time:** Evaluate the relaxation time due to star-star collisions for the following star systems: disk galaxy ($v \approx 30$ km/s, $n \approx 0.1\mathrm{pc}^{-3}$, size$\approx$ 30kpc), globular cluster ($v \approx 10$ km/s, $n \approx 10^4\mathrm{pc}^{-3}$, size$\approx$ 5pc), open cluster ($v \approx 1$ km/s, $n \approx 10\mathrm{pc}^{-3}$, size$\approx$ 5pc).

2. **Equations in cylindrical coordinates:** The natural coordinate system to work with for a disk galaxy is the cylindrical coordinates (r, ϕ, z) with the origin at the galactic center. z is the vertical coordinate parallel to the rotation axis and $z = 0$ represents the midplane of the disk. r is the radial coordinate perpendicular to z-axis and ϕ is the azimuthal coordinate.

   2.1 Given that $\mathbf{v} = v_r\hat{\mathbf{r}} + v_\phi\hat{\phi} + v_z\hat{\mathbf{z}} = \dot{r}\hat{\mathbf{r}} + r\dot{\phi}\hat{\phi} + \dot{z}\hat{\mathbf{z}}$, show that the equations of motion $\mathbf{a} = \dot{\mathbf{v}} = -\nabla\Phi$ in cylindrical coordinates read

$$\frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 = -\frac{\partial\Phi}{\partial r}, \tag{1}$$

$$\frac{d(r^2\dot{\phi})}{dt} = -\frac{\partial\Phi}{\partial\phi}, \tag{2}$$

$$\frac{d^2z}{dt^2} = -\frac{\partial\Phi}{\partial z}, \tag{3}$$

   where $\Phi(r, \phi, z)$ is the gravitational potential of the disk galaxy.

   2.2 Then show that the collisionless Boltzmann equation in cylindrical coordinates is given by eq(4-17) in the reference book by Binney & Tremaine (hereafter "BT").

   2.3 The moment equations in general coordinates were derived in class. Following the similar procedures, derive eqs(4-28), (4-29a), (4-29b), and (4-29c) from eq(4-17) in BT under the axisymmetric assumption (i.e. Φ and all average quantities do not vary with ϕ).

   2.4 Fluid equations in general coordinates were described in class. Derive the continuity and momentum equations in cylindrical coordinates:

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u_r) + \frac{1}{r}\frac{\partial}{\partial\phi}(\rho u_\phi) = 0, \tag{4}$$

$$\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial\phi} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r} - \frac{\partial\Phi}{\partial r}, \tag{5}$$

$$\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{u_\phi u_r}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial\phi} - \frac{1}{r}\frac{\partial\Phi}{\partial\phi}. \tag{6}$$

3. **Asymmetric drift:** As shown in class, in the steady state, the difference between the velocity of the dynamical LSR (local standard of rest) v_{LSR} and the average azimuthal velocity $\bar{v}_\phi$ of one stellar population in the solar neighborhood is related to the radial velocity dispersion $\bar{v_r^2}$:

$$v_a \equiv v_{LSR} - \bar{v}_\phi \propto -\bar{v_r^2}\frac{\partial n}{\partial r}. \tag{7}$$

The above relation for the asymmetric drift v_a may be realized as follows.

3.1 If all stars in the solar neighborhood have exact circular orbits, what is $\bar{v_r^2}$?

3.2 The orbits of old, metal-poor red dwarfs star are on average more elliptical than those of young, metal-rich stars. Which population of stars do you expect to have larger $\bar{v_r^2}$ and explain why based on your answer in Problem 3.1.

3.3 Let us assume that the galactic potential is Keplerian for simplicity[1]. Then consider the elliptical orbits of the hypothetical stars lying on the midplane of the disk galaxy with their apoceters/pericenters intersecting at the LSR. Look up any dynamics books and show that the azimuthal velocity v_ϕ at apocenter (pericenter) is slower (faster) than that of the LSR.

3.4 Explain qualitatively why $\partial n/\partial r$ of the hypothetical stars determines the sign of the asymmetric drift v_a.

4. **Free-fall time:** Find the free-fall collapse time of a non-rotating sphere of constant density ρ and radius R. (hint: starting with the Newton's law $d^2r/dt^2 = -GM/r^2$ with the initial inflow velocity $v = 0$, first show that $v(r) = -\sqrt{(8/3)\pi G\rho R^3(1/r - 1/R)}$. And then by further integration, you can obtain the free-fall time.

[1] Just for your information: as we already learned from the rotation curve in class, the potential is not Keplerian. Therefore the orbits of stars are not closed but precess (Chap. 3 in BT).