Green and Stokey (1983)

- The paper compares the relative efficiency of tournament and independent contract.
- Setting: One risk-neutral principal and n risk-averse agents. The output of an agent is affected by his own effort, individual noise, and a common noise.
- A special feature is that each agent can observe a private signal regarding the value of common shock.

- Main results:
    1. In the absence of common shock, independent contract dominates tournament.
    2. If the distribution of common shock is sufficiently dispersed, tournament dominates independent contract.
    3. If n is sufficiently large, then tournament dominates independent contract.

- Utility of agent i:

$$U(m_i, e_i) = u(m_i) - e_i;\ m_i \geq 0,\ e_i \geq 0,\ i = 1, \ldots, n;$$

where $u' > 0$, $u'' < 0$, m_i is income of i, and e_i is his effort level. Assume $u(m) \leq B$ for all m.

- Output:

$$y_i = z_i + \eta;$$

where $\eta \in R$ is a r.v. affecting all agents, and z_i is a r.v. with distribution function $F(\cdot; e_i)$ and density function $f(\cdot; e_i)$.

- Agent i observed a private signal $\sigma_i \in R$ about η before choosing e_i. Let $G(\eta, \sigma)$ be the joint distribution of η and $\sigma = (\sigma_1, \cdots, \sigma_n)$.
- The above formulation includes situations where all agents observe some signal, independent signals ($G(\eta, \sigma) = \prod_i G_i(\eta, \sigma_i)$), signals revealing η completely ($G(\eta, \sigma) = 1$ *iff* $\sigma_i = \eta \ \forall i$), or signals uncorrelated with η ($G(\eta, \sigma) = G_1(\eta) G_2(\sigma)$).
- z_i and (η, σ) are independent, and η has zero mean for all σ: $\int \eta dG(\eta, \sigma) = 0$.

- The principal observes only y_i's and maximizes profit:

$$\max \, E\Big[\sum_{i=1}^{n}(y_i - m_i)\Big].$$

- Under independent contract i's pay depends only on y_i; under tournament it depends on the order of y_i among all $y_j's$.

- For any wage function $R(y)$, let $v(y) \equiv u[R(y)]$, and let $\gamma = u^{-1}$.
- After observing σ_i, agent i chooses e_i to maximize $EU(R(y), e_i)$.
- Let the optimal decision rule of i be $X(\sigma_i)$.
- The principal thus chooses (v, X) to maximize expected profit, subject to the constraints that X be optimal, and the expected utility of the agent be at least $\underline{u}$.

- Let $S_{ci}(G)$ be the set of all feasible contracts for agent i:

$$S_{ci} \equiv \left\{ (v, X) \middle| v : R_+ \rightarrow [0, B], X : R \rightarrow R_+ \right\};$$

where

$$X(\sigma_i) =$$

$$\arg\max_e \int v(y) \int f(y - \eta; e) dG(\eta, \sigma_{-i} | \sigma_i) dy - e, \ \forall \sigma_i; \ \text{(IC1)}$$

$$\int \int \left[v(y) - X(\sigma_i) \right] f \left[y - \eta; X(\sigma_i) \right] dG(\eta, \sigma) dy \geq \underline{u}. \ \text{(IR1)}$$

- Let $P_{ci}(v, X, G)$ be the expected payoff of the principal from contract (v, X):

$$P_{ci}(v, X, G) \equiv \int \int \Big\{ y - \gamma\big[v(y)\big] \Big\} f\Big[y - \eta; X(\sigma_i) \Big] dydG(\eta, \sigma).$$

- $S_{ci}(G) \neq \emptyset$ since $(\underline{v} \equiv \underline{u}, X^0 \equiv 0) \in S_{ci}(G)$.
  Let P^0 be the payoff of the principal in this contract.
  That is, $P^0 \equiv P_{ci}(\underline{v}, X^0, G)$. P^0 is independent of G.

- Let $(W_1, W_2, \ldots, W_n)$ be the payoff structure, and let $w_i = u(W_i)$. W_i is the wage for the agent whose output ranked at the lowest ith position.
- $y_i \geq y_j$ iff $z_i \geq z_j$, which is independent of the value of η.
- Let ϕ_{jn} be the jth-order statistic of $(z_1, \ldots, z_n)$:

$$\phi_{jn}(z; e) = \frac{n!}{(n-j)!(j-1)!} f(z; e) F^{j-1}(z; e) \left[1 - F(z, e)\right]^{n-j}.$$

- The feasible set of tournament is

$$S_T(n; G) = \left\{ (w, \bar{e}) \middle| w \in [0, B]^n, \bar{e} \in R_+ \right\};$$

where

$$\bar{e} \in \arg\max_{e} \frac{1}{n} \sum_{j=1}^{n} w_j \int \frac{f(z; e)}{f(z; \bar{e})} \phi_{jn}(z, \bar{e}) dz - e; \text{ (IC2) and}$$

$$\frac{1}{n} \sum_{j=1}^{n} w_j - \bar{e} \geq \underline{u}. \text{ (IR2)}$$

- $S_T(n; G)$ is independent of G, so we write $S_T(n, G) \equiv S_T(n)$.
- The expected profit of the principal under $(w, \bar{e})$ is

$$
\begin{aligned}
&P_T(n, w, \bar{e}, G) \\
&\qquad \equiv \int \int y f(y - \eta; \bar{e}) dG(\eta, \sigma) dy - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_i) \\
&\qquad = \int z f(z, \bar{e}) dz - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_j) \\
&\qquad \equiv P_T(n, w, \bar{e}).
\end{aligned}
$$

- If there is no common shocks, i.e., if

$$\int_{\sigma \in R^n} dG(\eta, \sigma) = \begin{cases} 0 & \text{if } \eta < 0 \\ 1 & \text{if } \eta \geq 0, \end{cases}$$

then for any feasible tournament there is a feasible contract that dominates it. That is, if $G(\cdot)$ satisfies above, then given any $(w, \bar{e}) \in S_T(n)$ there exists $(v, X) \in S_{ci}(G)$ such that

$$P_{ci}(v, X, G) \geq P_T(w, \bar{e}, n).$$

Inequality is strict unless $(w, \bar{e}) = (\underline{u}, \underline{u}, \ldots, \underline{u}, 0)$.

- Proof: For any $(w, \bar{e}) \in S_T(n)$, Let (v, X) be such that

$$v(y) = \frac{1}{n} \sum_{j=1}^{n} w_j \phi_{jn}(y; \bar{x}) / f(y; \bar{x})$$

$$X(\sigma_i) = \bar{e}.$$

This contract gives each agent, for each y, the expected value of wages that would be given by the tournament contract $(w, \bar{e})$.

This contract satisfies (IC1) and (IR1):
The optimal effort of the agent under $v(\cdot)$ is

$$\arg\max_e \int v(z)f(z;e)dz - e$$
$$= \arg\max_e \int \frac{1}{n}\sum_{j=1}^{1} w_j \frac{\phi_{jn}(z,\bar{e})}{f(z,\bar{e})} f(z,e)dz - e.$$

Since $(w,\bar{e})$ satisfies (IC2), we know that the optimal effort must be $\bar{e}$, and (IC1) is satisfied. Moreover, since $(w,\bar{e})$ satisfies (IR2), we know that

$$\int v(z)f(z;\bar{e})dz - \bar{e} = \int \frac{1}{n}\sum_{j=1}^{n} \frac{\phi_{jn}(z;\bar{e})}{f(z;\bar{e})} w_j f(z;\bar{e})dz - \bar{e} \geq \underline{u}.$$

Finally,

$$
\begin{aligned}
P_c(v, X, G) &= \int \big(z - \gamma[v(z)]\big) f(z; \bar{e}) dz \\
&= \int z f(z; \bar{e}) dz - \int \gamma\big[\frac{1}{n} \sum_{j=1}^{n} \frac{\phi_{jn}(z; \bar{e})}{f(z; \bar{e})} w_j\big] f(z; \bar{e}) dz \\
&\geq \int z f(z; \bar{e}) dz - \frac{1}{n} \sum_{j=1}^{n} \gamma(w_j) \int \phi_{jn}(z, \bar{e}) dz \\
&= P_T(w, \bar{e}, n).
\end{aligned}
$$

The inequality comes from Jensen's inequality, as γ is strictly concave. If $w \neq (\bar{w}, \cdots, \bar{w})$, the inequality is strict. If $w = (\bar{w}, \cdots, \bar{w})$, then $\bar{e} = 0$. Moreover, unless $\bar{w} = \underline{u}$, otherwise the contract $(v \equiv \underline{u}, X \equiv 0)$ dominates $(w, \bar{e})$.

Obviously, this result implies that, in the absence of common error term, tournament is never optimal.

Comparison 2: With Common Error

- We are interested in contract vs. tournament as common term becomes diffuse.
- For this purpose, consider a sequence $\{G_k\}_{k=1}^{\infty}$ such that each G_k satisfies the properties of G, and

$$\int g_k(\eta, \sigma_{-i}|\sigma_i)\, d\sigma_{-i} \equiv g_{ki}(\eta|\sigma_i) < \frac{1}{k} \qquad (1)$$

for all η, σ and i; where g_k is density function of G_k.

- Proposition: Assume that $F_e(z; e)$ is of bounded variation in z, for all $e \geq 0$. Moreover, the bound M is uniform in e. Then there exists K such that for all $k > K$,

$$\max_{(w,\bar{e}) \in S_T(n)} P_T\big(w, \bar{e}, n\big) \geq \max_{(v,x) \in S_{ci}(G_k)} P_{ci}(v, X, G_k) \quad (2)$$

  for all i. The inequality is strict unless the LHS is P^0.
- (A function $f(\cdot)$ is of bounded variation on $[a, b]$ if $\int_a^b |f'(x)| dx < \infty$.)

- Proof: Let $\left\{(v_{ki}^*, X_{ki}^*)\right\}_{k=1}^{\infty}$ be a sequence of optimal contracts for agent i correspondence to $\left\{G_k\right\}_{k=1}^{\infty}$. If $X_{ki}^*(\sigma_i) > 0$, it must be that

$$\int v_{ki}^*(y) \int f_e\Big[y - \eta; X_{ki}^*(\sigma_i)\Big] g_{ki}(\eta|\sigma_i) d\eta dy = 1. \tag{3}$$

Since f_e is of bounded variation, (1) implies that

$$\begin{aligned} &\lim_{k\to\infty} \Big| \int f_e\Big(y - \eta; X_{ki}^*\Big) g_{ki}(\eta|\sigma_i) d\eta \Big| \\ &\leq \lim_{k\to\infty} \frac{1}{k} \Big| f_e\Big(y - \eta; X_{ki}^*\Big) \Big| d\eta \\ &\leq \lim_{k\to\infty} \frac{M}{k} = 0. \end{aligned}$$

Note that because $v_{ki}^*(y) \in \big[0, B\big]$ for all y, (3) cannot hold. Hence for k sufficiently large, $w_{ki}^* \equiv \underline{u}$ and $X_{ki}^* \equiv 0$, and therefore $P_c\big(v_{ki}^*, X_{ki}^*, G_k\big) = P^0$.

Since S_T and P_T are independent for G, the LHS of (2) is at least P^0 and is constant through out $\{G_k\}$.

Comparison 3: Large Number of Agents

- When the number of agents is large enough, not only does tournament dominate contract, but also optimal tournament approaches full-information solution.
- Proof is in three steps:

  1. Any contract for which the payoff function is piecewise continuous and the agent's optimal effort level is unique can be approtimated by a payoff function which is a step function and that changes the effort level chosen by agent in a continuous way.

  2. Every step-function contract can be approximated by a tournament with a sufficiently large number of agents.

  3. When number of agents is sufficiently large, optimal tournament approximates full information solution.

- Intuition : When $n \to \infty$, the realized distribution and density functions of outputs under e^* are almost $F(y - \eta; e^*)$ and $f(y - \eta; e^*)$.