Hölmström (1982)

- Two contributions:
  (1) Identifying free-riding as the main cause preventing efficiency in team production, and the role of the principal to break the budget balance constraint to overcome this problem.
  (2) The importance of relative performance evaluation to extract information, rather than to enhance competition or share risk, in order to improve efficiency in team production.

- A team consists of n agents, $i = 1, \ldots, n$.
- Effort level of every agent i, $e_i \in A_i \equiv R_+$, is unobservable.
- Cost of effort: $v_i(\cdot)$, with v_i', v_i'' >0, and $v_i(0) = 0$.
- Only joint output, $y : A \to R$, is observable, which is strictly increasing, concave, with $y(0) = 0$.

- Utility of agent i:

$$u_i(m_i, e_i) = m_i - v_i(e_i);$$

  where m_i is income.

- $s_i(y)$: Agent i's share of output y.
- Balance of budget requires that

$$\sum_{i=1}^{n} s_i(y) = y \quad \forall y. \tag{1}$$

- Output is a deterministic function of effort: $y = f(e)$.
- Assume that there exists e such that $f(e) - \sum_{i=1}^{n} v_i(e) > 0$.
- In equilibrium, every agent i chooses the level of e_i to maximize

$$s_i\big(y(e)\big) - v_i(e_i).$$

- FOC:

$$s_i'\big(y(e)\big)y_i(e) = v_i'(e_i); \quad \forall i \tag{2}$$

- Pareto optimality requires the effort, e^*, to satisfy

$$e^* = \arg\max_e \left(y(e) - \sum_{i=1}^{n} v_i(e_i) \right).$$

- FOC:

$$y_i(e^*) = v_i'(e_i^*); \quad \forall i. \tag{3}$$

- If e^* is an equilibrium, then (2) and (3) together implies that $s_i'(y) = 1 \ \forall i$.
- However, this contradicts with (1), since (1) implies that

$$\sum_{i=1}^{n} s_i'(y) = 1. \tag{4}$$

- **Theorem 1**: There does not exist a budget-balancing sharing rule which attain production efficiency as an equilibrium outcome in a team.
- Note that we've only proved Theorem 1 for differentiable sharing rules. For general case look up the paper.

- Any way out?
  (1) Hire a principal to monitor the agent. The principal must share output to give him incentive to monitor. (Alchian and Demsetz, 1972)
  (2) This paper proposes a simpler solution.
- Key: The budget-balancing constraint.
- Suppose the sharing rule is such that

$$\sum_{i=1}^{n} s_i(y) \leq y, \quad \forall y. \tag{5}$$

That is, no need to balance budget.

- **Theorem 2**: Without budget-balance, there exists a sharing rule in which e^* is the equilibrium effort level.
- Proof: Let

$$s_i(y) = \begin{cases} b_i(y) & \text{if } \; y \geq y(e^*) \\ 0 & \text{if } \; y < y(e^*). \end{cases} \tag{6}$$

  The value of b_i's are chosen so that $b_i > v_i(e_i^*)$ and $\sum_{i=1}^n b_i = y(e^*)$. This is possible because $y(e^*) - \sum_{i=1}^n v_i(e^*) > 0$. Obviously agent i's best response is $e_i = e_i^*$. (Note that $s_i(y)$'s do not always add up to y.)
- The role of (5), that is, the assumption that budget is not required to balance, is to allow group penalty that deters free-riding.

- Let $f(y;\, e)$ be the density function of output, with $F(\cdot)$ its distribution function.
- We can still prove inefficiency of equilibrium when budget has to be balanced:

  Pareto optimality requires that

  $$e_i \in \arg\max_{e_i'} \int yf(y; e_i', e_i)dy - \sum_{i=1}^{n} v_i(e_i');$$

  whose FOC requires

  $$\int yf_{e_i}(y; e)dy - v_i'(e_i) = 0, \quad \forall i.$$

  By (1) we know

  $$\sum_{i=1}^{n} \int s_i(y) f_{e_i}(y; e)dy - v_i'(e_i) = 0. \tag{7}$$

- However, every agent i choose e_i to maximize

$$\int s_i(y) f(y; e) dy - v_i(e_i),$$

whose FOC requires

$$\int s_i(y) f_{e_i}(y; e) dy - v_i'(e_i) = 0 \quad \forall i.$$

This contradicts (7).

- Any way out?
  Assume that

  A 1: F is convex in e.

  A 2: $\lim_{y \to -\infty} F_i/F = -\infty$.
- **A2** means that at a very low output, change in effort is easy to detect by checking output.

- **Theorem 3**: Assuming A1 and A2, Pareto optimum can be arbitrarily approximated by using group penalty.
- Proof: Let

$$s_i(y) = \begin{cases} s_i y & , \text{ if } \ y \geq \bar{y} \\ s_i y - k_i & , \text{ if } \ y < \bar{y}; \end{cases} \tag{8}$$

where $k_i > 0$, $\sum s_i = 1$, and $\bar{y}$ is a fixed output. In order that e^* be a equilibrium effort level, FOC requires that, for all i,

$$s_i \int y f_{e_i}(y; e^*) dy - k_i F_{e_i}(\bar{y}; e^*) - v_i'(e_i^*) = 0. \tag{9}$$

A1 guarantees (9) is necessary and sufficient for e^* to be equilibrium.

For any fixed $\bar{y}$ choose k_i so that (9) holds.

There always exists k_i to satisfy (9) because $\int y f_{e_i}(y;e^*)dy > 0$ and $F_{e_i} \leq 0$. (Both by FOSD of $f(\cdot)$.)

Note that the expected loss of the team under this sharing rule is $\sum_i k_i F(\bar{y}, e^*)$. From (9) we know that

$$k_i = \frac{\int s_i y f_{e_i}(y;e^*)dy - v_i'(e_i^*)}{F_{e_i}(\bar{y};e^*)} \equiv \frac{A_i}{F_{e_i}(\bar{y};e^*)}.$$

Let $\bar{y}$ decreases while adjusting k_i so that (9) always holds. The loss is

$$\sum_{i=1}^{n} \frac{A_i F(\bar{y};e^*)}{F_{e_i}(\bar{y};e^*)};$$

which, by A2, approaches zero when $\bar{y}$ decreases.

- Intuition: Make "target" $\bar{y}$, easy to meet, while setting stiff punishment to prevent shirk, which is easy to detect at low level of $\bar{y}$ (A2).

Adding role of Principal I: Sufficient Statistic

- Since principal is needed to break budget constraint, suppose there is indeed a risk-neutral principal and n risk-averse agents.
- Suppose, in addition to y, a vector of signal x is observed (and verifiable). This x may or may not contain y.

- Let $G(x;e)$ be distribution function of x given e, with $g(x;e)$ its density function.
- Principal's problem:

$$\max_{e,s_i(x)} \int \big(E(y|x;e) - \sum_i s_i(x)\big) g(x;e)dx,$$

$$s.t. \int u_i\big(s_i(x)\big) g(x;e)dx - v_i(e_i) \geq \underline{u_i},$$

$$e_i \in \arg\max_{e_i'} \int u_i\big(s_i(x)\big) g(x;e_i',e_i)dx - v_i(e_i'),$$

where $E(y|x;e)$ is the expected value of y given x and e.

- Definition (extension of Hölmström 1979): A function $T_i(x)$ is sufficient for x with respect to e_i, if there exists functions $h_i(\cdot) \geq 0$ and $p_i(\cdot) \geq 0$ such that

$$g(x; e) = h_i(x; e_{-i}) p_i(T_i(x), e). \tag{10}$$

- Note that $h_i(x; e_{-i})$ does not change with e_i. Also, as long as the value of $T_i(x)$ remains the same, variations in x does not change the value of $p_i(\cdot)$.
- $T(x) = \big(T_1(x), \ldots, T_n(x)\big)$ is sufficient for x w.r.t. e if $T_i(x)$ is sufficient for e_i $\forall i$.

- **Theorem 4** Let $T(x)$ be sufficient for x w.r.t. e. Then given an incentive scheme $\{s_i(x)\}$, there exists $\{\hat{s}_i(T_i)\}$ which weakly Pareto dominates it.
- Intuition: If $T_i(x)$ is sufficient for x w.r.t. e_i, then paying agent i by the values of $T_i(x)$, rather than x, loses nothing in information. Moreover, if agent i is paid the fixed average utility whenever x falls into a region of x's whose $T_i(x)$ are same, then wage bill will be smaller since agent is risk averse.

Proof. Consider the case of a single agent. Define $\hat{s}(T)$ so that

$$\begin{aligned} u\big(\hat{s}(T)\big) &= \int_{T(x)=T} \frac{u\big(s(T)\big)g(x,e)dx}{p(T,e)} \\ &= \int_{T(x)=T} u\big(s(x)\big)h(x)dx. \end{aligned} \tag{11}$$

(11) implies that, for all e, the agent has the same expected utility under $s(x)$ and $\hat{s}(T)$.

The agent will therefore exert same effort under $s(x)$ and $\hat{s}(T)$.

From Jensen's inequality, (11) implies

$$\int \hat{s}\big(T(x)\big)g(x,e)dx \le \int s(x)g(x,e)dx.$$

The principal's wage payment is therefore less under $\hat{s}(T)$ than $s(x)$.

The case with more than one agent is identical, as other agent's effort can be inferred from equilibrium.

- Define $T(x)$ to be sufficient at e if for all i and T_i,

$$\frac{g_{e_i}(x_1, e)}{g(x_1, e)} = \frac{g_{e_i}(x_2, e)}{g(x_2, e)} \text{ for all } x_1,\, x_2 \in \{x|\, T_i(x) = T_i\}. \tag{12}$$

- Obviously, (12) is implied by (10). Moreover, (12) implies (10) if it holds for all e.
- Definition: $T(x)$ is globally sufficient if (12) is true for all e and i; and is globally insufficient if for some i, (12) is false for all e.

- **Theorem 5** Assume $T(x)$ is globally sufficient for x. Let $\{s_i(x) = \hat{s}_i(T(x))\}$ be a nonconstant incentive scheme. Then there exists sharing rule $\{\bar{s}_i(x)\}$ which strictly Pareto dominates $\{s_i(x) = \hat{s}_i(T(x))\}$. Moreover, $\{\bar{s}_i(x)\}$ can be chosen to induce the same equilibrium effort as $\{s_i(x)\}$.

  Proof: See paper.

Adding the Role of Principal II: Relative Performance Evaluation

- Suppose information system is richer so that individual performances can be observed.
- Specifically, assume

$$y(e,\theta) = \sum_i y_i(e_i,\theta_i).$$

- Outputs are said to be independent if

$$f(y,e) = \prod_{i=1}^{n} f_i(y_i,e_i),$$

which implies that $T_i(y) = y_i$ is sufficient for y w.r.t. e_i.

- **Theorem 6** Suppose y_i are monotonic in θ_i for all i. Then the optimal sharing rule for agent i depends on i's output alone iff outputs are independent.

Proof. Sufficiency: Since $T_i(y) = y_i$ is sufficient for y w.r.t. e_i, the optimal sharing rule s_i depends on y_i alone.

Necessity: Suppose, say, θ_1 and θ_2 are dependent. Without loss of generality we can assume $y_2 = \theta_2$. Keeping e_2 constant at the equilibrium value, the joint density of y_1 and $y_2(= \theta_2)$, conditional on e_1, is

$$f(y_1, \theta_2, e_1) = \tilde{f}\left(y_1^{-1}(e_1, y_1), \theta_2\right),$$

where y_1^{-1} is the inverse of $y_1(e_1, \theta_1)$, and $\tilde{f}(\theta_1, \theta_2)$ is joint density for $(\theta_1,\ \theta_2)$.

$$\frac{f_{e_1}(y_1, \theta_2, e_1)}{f(y_1, \theta_2, e_1)} = \frac{\tilde{f}_1\left(y_1^{-1}(e_1, y_1), \theta_2\right)}{\tilde{f}\left(y_1^{-1}(e_1, x_2), \theta_2\right)} \frac{\partial y_1^{-1}(e_1, y_1)}{\partial e_1}.$$

Since θ_1 and θ_2 are dependent, $\tilde{f}_1/\tilde{f}$ depends on θ_2. Therefore, (12) does not hold, and Theorem 5 implies that agent i's sharing depends both on y_1 and y_2.