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Response in kinetic Ising model to oscillating magnetic fields

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Abstract

Ising models obeying Glauber dynamics in a temporally oscillating magnetic field are analyzed. In the context of stochastic resonance, the response in the magnetization is calculated by means of both a mean-field theory with linear-response approximation, and the time-dependent Ginzburg–Landau equation. Analytic results for the temperature and frequency dependent response, including the resonance temperature, compare favorably with simulation data. © 1998 Elsevier Science B.V.

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1. Introduction

The Ising model with Glauber dynamics in an oscillating magnetic field was recently considered with Monte Carlo (MC) simulations in Refs. [1,2]. The phenomenon of stochastic resonance (see, e.g., Ref. [3]) was explored by viewing the Ising model as a system of coupled two-state oscillators, driven by the oscillating field and "noises" which are taken to be thermal fluctuations. The phenomenon was revealed by a characteristic peak in the correlation function $C(T)$ between the magnetic field and the magnetization $M(t)$ versus the temperature T of the system. The resonance temperature T_r (the temperature at which $C(T)$ has a maximum) was systematically computed as a function of the driving period, lattice size and driving amplitude, both for two-dimensional (2D) [1] and three-dimensional (3D) [2] systems. The one-dimensional (1D) case was analyzed by Brey and Prados [4] within linear response theory.

The present work is a natural continuation of those studies, considering analytically the 2D and 3D cases. We will present two approaches. The mean-field theory with linear response approximation will be discussed first. Then in 2D where the mean-field theory is not as good as in other dimensions, a more refined time-dependent Ginzburg–Landau (TDGL) approach will be presented, with significant improvements.

Recently, kinetic Ising systems in oscillating external fields have also been examined both experimentally and theoretically in Ref. [5]. The focus was on properties below the zero-field critical point, such as the frequency dependence of the probability distributions for the hysteresis-loop area and the residence time. The latter quantity for small systems in moderately weak fields suggests further evidences of stochastic resonance. Very recently, finite-size effects versus driving frequency have been analyzed as a dy-

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namical critical phenomena [6]. In contrast to these works, ours is focused on the temperature dependence above the zero-field critical point.

Stochastic resonance is conventionally studied by means of the signal-to-noise ratio (see, e.g., Ref. [3]). For small magnetic field, this quantity has been obtained for the Ising model from the power spectrum of the magnetization, exactly in 1D [7] and in higher dimensions by simulations and mean-field approaches [8]. The general result is that this ratio exhibits a peak at a definite temperature above T_c, weakly dependent on the driving frequency.

2. Mean-field theory and linear-response approximation

Our starting point is the master equation for the kinetic Ising model obeying Glauber dynamics [9],

$$P(\sigma;t+1) - P(\sigma;t) = \sum_{\sigma'} [w(\sigma' \to \sigma)\, P(\sigma';t) - w(\sigma \to \sigma')\, P(\sigma;t)], \tag{1}$$

where $P(\sigma;t)$ is the joint probability of finding the spin configuration σ at time t, and the w are the transition rates between two configurations which differ by one spin flip. For the heat-bath algorithm, the rate function is chosen as

$$w(\sigma \to \sigma') = \frac{1}{1 + \mathrm{e}^{-\beta[E(\sigma)-E(\sigma')]}},$$

with $\beta = 1/T$ (hereafter the Boltzmann constant $k \equiv 1$), and $E(\sigma)$ is the energy of σ in a magnetic field h,

$$E(\sigma) = -J \sum_{\mathrm{nn}} S_i S_j - h(t) \sum_i S_i, \tag{2}$$

where $h(t) = A\sin(\omega t)$ and $\sum_{\mathrm{nn}}$ denotes a summation over nearest neighbors in a square or cubic lattice.

Let us denote the configuration σ by the values of the spins $S_1, S_2, \ldots, S_V$, with system volume given by $V = N^d$. d is the spatial dimension of the system and N is its linear size. Since $S_i = \pm 1$, it is easy to rewrite (1) as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} P(S_1, S_2, \ldots, S_V; t) &= -\sum_{j=1}^{V} w_j(S_j)\, P(S_1, S_2, \ldots, S_V; t) \\ &\quad + \sum_{j=1}^{V} w_j(-S_j)\, P(S_1, S_2, \ldots, -S_j, \ldots, S_V; t)\end{aligned} \tag{3}$$

with

$$\begin{aligned} w_j(S_j) &= \tfrac{1}{2}[1 - S_j \tanh(E_j/T)], \\ E_j &= J \sum_{k=1}^{z} S_k + h, \end{aligned} \tag{4}$$

where the last sum runs over the z nearest neighbors of the spin S_j, with $z = 2d$. Multiplying both sides of (3) by S_l and performing an ensemble average (denoted by $\langle\ \rangle$), after some simple mathematical tricks, we get the basic equation for the Glauber dynamics,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S_l\rangle = -\langle S_l\rangle + \langle \tanh(E_l/T)\rangle. \tag{5}$$

Invoking the mean-field approximation, we replace E_l by $Jz\langle S\rangle + h$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle S\rangle = -\langle S\rangle + \tanh[(h + T_c^{\mathrm{MF}}\langle S\rangle)/T], \tag{6}$$

where $T_c^{\mathrm{MF}} = Jz$ is the mean-field critical temperature. In the absence of h, the magnetization is given by the stationary solution of the well-known equation,

$$\langle S\rangle_0 = \tanh[T_c^{\mathrm{MF}}\langle S\rangle_0/T]. \tag{7}$$

For small $h(t)$, we may use the linear-response theory in (6) by first writing $\langle S\rangle(t) = \langle S\rangle_0 + \Delta S(t)$ and considering the $h/T \ll 1$ and $\Delta S/T \ll 1$ limits. Performing the Taylor expansion and keeping only the first-order terms, Eq. (6) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta S = -\frac{\Delta S}{\tau_{\mathrm{MF}}} + \frac{A}{T}(1 - \langle S\rangle_0^2)\sin(\omega t), \tag{8}$$

where

$$\tau_{\mathrm{MF}} = \frac{1}{1 - (T_c^{\mathrm{MF}}/T)(1 - \langle S\rangle_0^2)} \tag{9}$$

is the relaxation time. The solution can be found easily,

$$\Delta S(t) = \Delta S_0 \sin(\omega t - \theta_{\mathrm{MF}}), \tag{10}$$

with the phase shift and amplitude given by

$$\theta_{\rm MF} = \arctan(\omega\tau_{\rm MF}), \tag{11}$$

$$\Delta S_0 = \frac{A}{T}(1-\langle S\rangle_0^2)\frac{1}{\sqrt{1/\tau_{\rm MF}^2+\omega^2}}. \tag{12}$$

The correlation function between the total magnetization $M = V\langle S\rangle$ and the external field $h(t)$ can be computed,

$$\begin{aligned} C = \overline{M(t)\,h(t)} &\equiv \frac{V\omega}{2\pi}\int_0^{2\pi/\omega} \Delta S(t)\, h(t)\,\mathrm{d}t \\ &= \frac{VA^2}{2T}(1-\langle S\rangle_0^2)\frac{\tau_{\rm MF}}{1+\omega^2\tau_{\rm MF}^2}. \end{aligned} \tag{13}$$

Here the overline denotes a temporal average over a period $P = 2\pi/\omega$. In the $T > T_c^{\rm MF}$ domain, $\langle S\rangle_0 = 0$, thus C becomes

$$C_{T>T_c^{\rm MF}} = \tfrac{1}{2}VA^2\frac{T-T_c^{\rm MF}}{(T-T_c^{\rm MF})^2+\omega^2T^2}. \tag{14}$$

3. Time-dependent Ginzburg–Landau approach

Before comparing (13) to simulations, we present an alternative, continuum approach to compute C. For an Ising system with non-conservative order parameter (model A [10]), the time-dependent Ginzburg–Landau (TDGL) equation for the local magnetization density $\phi(\boldsymbol{r},t)$ takes the following form,

$$\frac{\partial\phi}{\partial t} = -\Gamma\frac{\delta\mathcal{H}}{\delta\phi}+\zeta, \tag{15}$$

$$\mathcal{H} = \int \mathrm{d}\boldsymbol{r}\,\left(\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}u\phi^2+\frac{g}{4!}\phi^4\right), \tag{16}$$

where $\mathcal{H}$ is the coarse grained Hamiltonian. For our present purpose, the white noise $\zeta(\boldsymbol{r},t)$ which accounts for the effect of thermal fluctuations is irrelevant. Conventionally, parameters Γ, u and g in (16) are understood to be obtained by coarse graining the microscopic dynamics (1). For critical properties, the sole important temperature dependence in these parameters lies in $u \propto T - T_c^{\rm GL}$, giving rise to the spontaneous symmetry breaking below the critical temperature $T_c^{\rm GL}$. For our purposes of comparing with simulations, more precise dependences on T are required. To this end, we outline here a refined mean-field approach in the continuum limit. The same approach has been successfully applied to the two-species driven diffusive systems [11]. This approximation is expected to be good outside the critical region. However, this turns out to be not a serious handicap because the presence of an oscillating field prevents the system from building up critical correlations.

In a mean-field approximation, the joint probabilities in (1) are factorized into singlet probabilities $p(\boldsymbol{r};t)$ for finding the spin up at site $\boldsymbol{r}$ at time t. Since a spin flip depends on a total of $z+1$ spins in (1), the factorization effectively produces a series expansion of $\mathcal{H}$ in powers of ϕ up to ϕ^{z+1}. This is followed by the continuum limit, i.e., expansions in the derivatives such as

$$p(x\pm1,y;t) \to p(x,y;t) \pm \frac{\partial p(x,y;t)}{\partial x} + \tfrac{1}{2}\frac{\partial^2 p(x,y;t)}{\partial x^2} + \ldots$$

For long-distance behavior, we stop at the order as shown, consistent with (16). By identifying p as $(\phi+1)/2$ and collecting terms according to powers of ϕ, we obtain from (1) a kinetic equation for ϕ after some algebra. For $h = 0$, we find precisely the deterministic part of (15) with

$$\Gamma = \tfrac{1}{8}(-2W_4+2W_{-4}-W_8+W_{-8}), \tag{17}$$

$$u = \frac{1}{8\Gamma}(6W_0+12W_4-4W_{-4}+5W_8-3W_{-8}), \tag{18}$$

$$g = \frac{3}{2\Gamma}(-6W_0-4W_4+4W_{-4}+5W_8+W_{-8}), \tag{19}$$

where $W_n \equiv 1/(1+\mathrm{e}^{n\beta J})$ contains the desired explicit T dependence. The coefficient for ϕ^5 happens to vanish for heat-bath rates. When a small uniform field h is applied, to $\mathrm{O}(h)$ we have finally the deterministic kinetic equation

$$\frac{\partial\phi}{\partial t} = -\Gamma(-\nabla^2\phi+u\phi+\tfrac{1}{6}g\phi^3-\mu h), \tag{20}$$

where $\mu = \beta(3W_0^2+4W_4W_{-4}+W_8W_{-8})/2\Gamma$. It is useful to note that Γ, g and μ in (20) are positive definite for all T, whereas u has one zero at $T_c^{\rm GL} \approx 3.0901J \approx 1.3618T_c$, where $T_c = -2J/\ln(\sqrt{2}-1) \approx 2.2692J$ is exact. This is an improvement over $T_c^{\rm MF} = 4J$ from the last section. Moreover, we reproduce the first few terms of the high-temperature series expansions of thermodynamic quantities such as the susceptibility and the relaxation time. In the $\beta \to 0$ limit, we recover the mean-field results of the last section: $u \approx 1/\beta J - 4$, $\Gamma \approx \beta J$, $g \approx 48(\beta J)^2$, and $\mu \approx 1/J$.

For small h and $T > T_c^{\mathrm{GL}}$, the nonlinear term $g\phi^3$ in (20) is negligible. The total magnetization $M(t) = \int \mathrm{d}\boldsymbol{r}\, \phi(\boldsymbol{r}, t) = \tilde{\phi}(\boldsymbol{q} = 0, t)$ in response to an external field can then be computed easily, where $\tilde{\phi}$ denotes the spatial Fourier transform of ϕ. It satisfies $\partial M/\partial t = -\Gamma u M + \Gamma \mu \tilde{h}(\boldsymbol{q} = 0, t)$. We readily find

$$M(t) = \frac{V\mu A\Gamma}{\sqrt{(\Gamma u)^2 + \omega^2}} \sin(\omega t - \theta_{\mathrm{GL}}), \tag{21}$$

where the phase shift is $\theta_{\mathrm{GL}} = \arctan(\omega/\Gamma u)$. The correlation function with h is then given by

$$C_{T>T_c^{\mathrm{GL}}} = \frac{VA^2\Gamma^2\mu u}{2[(\Gamma u)^2 + \omega^2]}. \tag{22}$$

Note that this coincides with the mean-field result (14) in the high-temperature limit.

For $T < T_c^{\mathrm{GL}}$, the term proportional to g is needed to break the symmetry, leading to the spontaneous magnetization $m = \sqrt{-6u/g}$ (recall that $g > 0$ for all T, and $u < 0$ for $T < T_c^{\mathrm{GL}}$.) Linearizing about m, we find precisely the same form of C as $T > T_c^{\mathrm{GL}}$ except that u is replaced by $-2u$ in (22).

Examining (20), one may ask why one should expect stochastic resonance above T_c where the potential has a single well. Besides, C is computed without ever using the noise term ζ in (15). The resolution of these apparent contradictions with conventional stochastic resonance lies in the fact that thermal effects, regarded as the "noises" here, have been separated for mathematical convenience into a deterministic and a stochastic part in (15). Essentially, the deterministic part (the entropic effect) has been incorporated with the two-state nature of the spins, resulting in a single-well free energy functional, whereas ζ accounts for the remaining stochastic part. Hence, our analysis is based on a transformed description in which part of the noises are integrated with the double-well potential. We are not aware of a similar formulation in conventional studies of stochastic resonance.

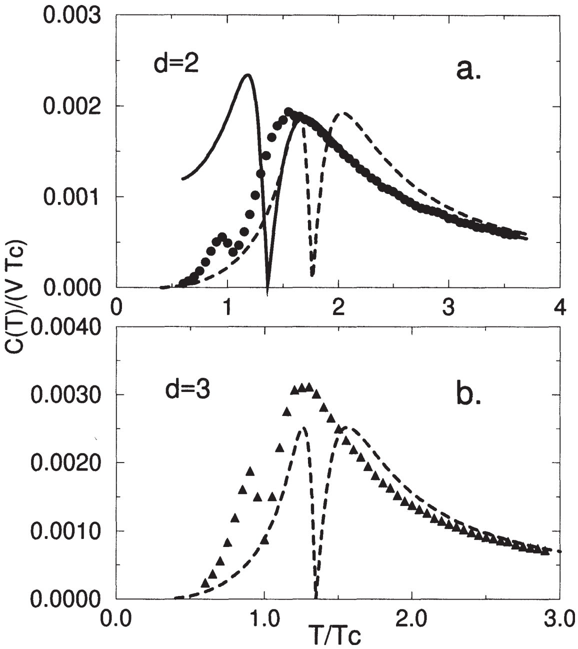

Fig. 1. $C(T)/VT_c$ versus temperature for $P = 40$ and $A = 0.05T_c$ for 2D in (a) and 3D in (b). Dots are MC simulation results in 2D ($N = 200$), triangles are MC simulations in 3D ($N = 40$), the continuous line is from the TDGL approximation and the dashed line is the mean-field result.

4. Discussion and comparison with simulations

From the simulation data in Refs. [1,2], we learn that the system has a maximum response to external driving at a definite temperature T_r which depends on the driving frequency. Hence T_r can be designated as the *resonance temperature*. From the analytically determined correlation functions in (14) and (22), we find two peaks in C above and below the respective T_c, and also $C(T_c) = 0$, as shown in Fig. 1. This double-peak structure in C is consistent with simulations for larger lattice sizes (up to $N = 200$ for 2D and $N = 40$ for 3D) and with smaller steps in T than reported in Refs. [1,2]. The reason for missing the peak below T_c in our earlier simulations may be the use of small lattice sizes. Note that the peak below T_c is much smaller than the one above and its position is less sensitive to the driving period. The reason for the overestimated theoretical values of the peaks below T_c may the frustration of the system to order in the presence of $h(t)$. Such frustration probably arises from nucleation of droplets of the stable phase inside the metastable phase [5]. Such local excitations have not been taken into account in our calculations. Instead, a uniform response of the system about one of the two local minima below T_c has been assumed.

We believe that this also explains the discrepancy at T_c, where simulations show a small but finite $C(T)$. Finite-size effects are not of great concern here because, as mentioned above, the correlation length even at T_c is truncated by h. In simulations, we have checked the convergence in $C(T)$ for $N \geqslant 50$ in 2D.

Focusing on $T > T_c$ from now on, the TDGL predictions for $C(T)$ are more accurate than those of the mean-field theory in general. They both converge to the simulations in the tails at $T \gg T_c$ (see Fig. 1). In 3D the mean-field theory is already acceptable except for the peak position, which is affected by the inaccuracy of $T_c^{\rm MF}$.

Turning our attention to the amplitude dependence, replotting the simulation data from Refs. [1,2] suggests that the height of the peak $C(T_r) \propto A^2$, in agreement with (14) and (22). For not too large frequencies and small A, the theoretical proportionality constant agrees well with simulations. For example, the slope of $C(T_r)/VT_c$ versus A^2/T_c^2 for $P = 50$ in 2D gives 0.92 from simulations [1], 0.96 from TDGL and 0.99 from mean-field approach. In 3D the same slope is 0.88 from simulations [2], and 1.29 from mean-field approach (In 3D the comparison are worse because T_r is much closer now to T_c.) This proportionality is a manifestation of the linear response of the system to h, which breaks down at large enough amplitudes. Our new simulations show that this happens for $A/T_c > 0.15$ in 2D for $P = 40$.

A quantity of significant interest is the resonance temperature $T_r(P)$. It can be determined analytically from (14)

$$T_r^{\rm MF} = T_c^{\rm MF}\left(1 + \sqrt{1 - \frac{1}{\omega^2 + 1}}\right), \tag{23}$$

and numerically from (22) for $T_r^{\rm GL}$. These together with simulation results are presented in Fig. 2. The agreements are reasonable. As expected the mean-field approximation is quite good in 3D but in 2D the TDGL approximation is better.

The results in Fig. 2 confirm the earlier observation in Refs. [1,2] that for $P \to \infty$ we get $T_r \to T_c$. This result is also consistent with the one obtained

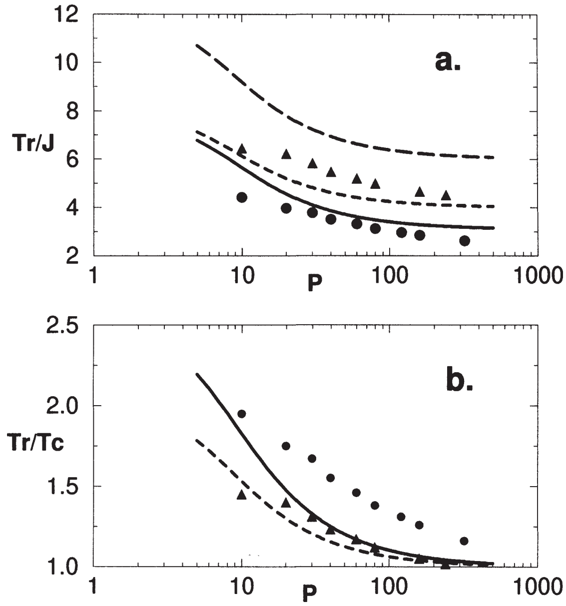

Fig. 2. Resonance temperature above T_c versus driving period P for $A = 0.05$, on absolute scale T_r/J in (a) and on relative scale T_r/T_c in (b). The long-dashed and short-dashed lines in (a) are the mean-field results for 3D and 2D respectively, in the rest the symbols mean the same as in Fig. 1.

by Brey and Prados [4] in 1D where the above limit becomes $T_r \to T_c = 0$. In the opposite limit $P \to 1$ (in unit of Monte Carlo steps $P \geqslant 1$) both the theory in 1D [4] and our approximations in 2D and 3D suggest $T_r \to$ const. Unfortunately, in Refs. [1,2] the wrong conclusion $T_r \to \infty$ was drawn in this limit. Similarly, the position of the peak below T_c also converges to T_c in the $P \to \infty$ limit.

In passing, we also derive [8] the relationship between the correlation function and the hysteresis-loop area $\mathcal{A}$,

$$\mathcal{A} = 2\pi C \tan\theta, \tag{24}$$

where θ is the phase shift between h and M. This result has also been derived recently by Acharyya [12], and relates our results of C to that of $\mathcal{A}$ as observed in Ref. [5].

5. Conclusions

Using mean-field with linear-response and TDGL approximations, the characteristics of the resonance peaks observed in kinetic Ising models in oscillating magnetic fields [1,2] are reproduced. New simulations improve earlier results by confirming the analytically predicted double peaks. Focusing mostly on the behavior above T_c (where our approaches work better), we determine the dependence of the resonance temperature as a function of driving frequency and amplitude. We confirm the already predicted result in Refs. [1,2] that $T_r \to T_c$ for the limit of practically interesting driving frequencies ($P \to \infty$), and corrected the wrong extrapolation in the opposite limit $P \to 1$. We introduce a refined TDGL approach which improves significantly the mean-field results in 2D, but in 3D the mean-field approximation is already acceptable. We have thus demonstrated that the stochastic resonance in kinetic Ising models above T_c can be understood by means of rather simple theoretical approaches for small driving amplitudes.

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