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Phase transition in a spring-block model of surface fracture

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Abstract. – A simple and robust spring-block model obeying threshold dynamics is introduced to study surface fracture of an overlayer subject to stress induced by adhesion to a substrate. We find a novel phase transition in the crack morphology and fragment-size statistics when the strain and the substrate coupling are varied. Across the transition, the cracks display in succession short-range, power law and long-range correlations. The study of stress release prior to cracking yields useful information on the cracking process.

Apart from its significance in industrial applications, cracking of a brittle material contains interesting physics and has attracted a lot of attention recently [1]. Two kinds of ongoing research can be identified: the first deals with the dynamic instability in the propagation of a single crack [2], and the second is concerned with the collective behavior of many interacting cracks, such as the global pattern and distribution of fragments [3]-[5]. The latter receives less attention and will be our focus here.

Although crack patterns in nature occur in a variety of contexts [6] over a wide range of sizes from the millimeters on a monolayer of packed polystyrene spheres [3] to the kilometers of giant crack networks on playas, they are formed by the same basic processes. Typically, when an overlayer dries and shrinks, adhesion to a substrate resists shrinkage and induces internal stress. The stress may be released by slipping at the interface of contact or by cracking the overlayer; their competition leads to the variety of crack patterns. In recent experiments, the scale and geometry of patterns were found to depend on the strength of adhesion, the thickness of the layer and boundary conditions [4], [5]. Analytical approaches are difficult and scarce, due to the complexity of multi-crack interactions [7]. Thus, simulations using simple models are expected to provide useful information [3], [8], [9].

In this letter, we present a spring-block model of fracture for a two-dimensional overlayer in contact with a substrate. Our goal is to identify the major control parameters and to explore various statistical properties of fracture not readily accessible to experiments.

Model. – Our model consists of a square array of blocks, interconnected among nearest neighbors by coil springs with spring constant K and relaxed spring length l. To compare with experiments, free boundary conditions are used, with three nearest neighbors on edges and two at corners. Initially, the blocks are randomly displaced about their mean positions

$\mathbf{r}_0 = (ia, ja)$ by (x, y) on a rough substrate, where $i, j = 1, \ldots, L$, and a is the lattice constant. Motivated by the drying process described above, internal stress may be introduced in various ways: a may be fixed but l is decreased in time to model contraction; or both a and l may be fixed to impose an initial tensile strain ($s = (a - l)/a > 0$), while K is increased to model material stiffening [9]. As a first study, we will adopt the simpler second approach here.

Contrary to similar models of earthquakes [10], [11], in which the stress arises from relative motion of two surfaces and released by block slips alone, the stress here is imposed uniformly by increasing K and released by block slips *and* spring breaks via a threshold dynamics: a block slips to a force-free position if the net force from its neighbors $F > F_{s+}$ (the threshold for slipping), and a spring breaks ($K \equiv 0$) if the tension $b > F_c \equiv \kappa F_s$ (the threshold for cracking). With all forces initially below thresholds, K is increased slowly with F_s and F_c fixed. When either threshold is reached, stress is dissipated and redistributed accordingly, causing further slips and/or cracks until $F < F_s$ and $b < F_c$ everywhere. This constitutes one *event*. K is increased only between individual events, corresponding to slow drying where the rate of driving is infinitesimal compared to that of relaxation.

The force exerted on a given block at $\mathbf{r} = \mathbf{r}_0 + (x, y)$ by a neighboring block at $\mathbf{r}'$ is given by $(|\mathbf{r} - \mathbf{r}'| - l)K$. Since cracking is primarily due to tensile stresses, this non-linear dependence on the coordinates leads to unnecessary complications in updating the configurations. To simplify and compare with previous models, we expand the forces to first order in (x, y) to obtain the force components on a block at (i, j) (cf. [11])

$$\begin{cases} F_x = (a - l + x_{i+1,j} - x_{i,j})K_1 + (x_{i,j+1} - x_{i,j})sK_2 + \\ \qquad +(-a + l + x_{i-1,j} - x_{i,j})K_3 + (x_{i,j-1} - x_{i,j})sK_4\,, \\ F_y = (y_{i+1,j} - y_{i,j})sK_1 + (a - l + y_{i,j+1} - y_{i,j})K_2 + \\ \qquad +(y_{i-1,j} - y_{i,j})sK_3 + (-a + l + y_{i,j-1} - y_{i,j})K_4, \end{cases} \tag{1}$$

where the subscript of K denotes different springs. Likewise, the tension of the spring between two blocks at (i, j) and $(i + 1, j)$ has the components $b_x = (a - l + x_{i+1,j} - x_{i,j})K$ and $b_y = (y_{i+1,j} - y_{i,j})sK$.

Without loss of generality, we hereafter choose our units such that $a = 1 = F_s$. Of the remaining parameters [12] $\{s, \kappa, K\}$, K may be used to define the "time" $t \equiv K$. Note that a larger κ means either a stronger material (larger F_c) or a weaker substrate coupling (smaller F_s).

Results. – We have simulated the model for a wide range of values of the dimensionless parameters: $0.01 \le s \le 0.98$, $1 \le \kappa \le 30$, and system sizes $20 \le L \le 300$. Clearly, in the slippery limit $\kappa = \infty$, no spring is broken and our model becomes a variant of previous slip-stick models [11], [10] in their conservative limits, with almost the same slip-size statistics. However, notice that our model is driven multiplicatively by K, whereas previous models are driven by an additive force term. For finite κ, the system contracts by block slips before it cracks into disjoint fragments. In agreement with intuition and experiments [4], [8], both the waiting time t_c for the first spring to break and the mean fragment size increases with κ (see fig. 1 and below).

Figure 1 also reveals two kinds of crack patterns, depending on an s-dependent ratio κ^* for large s:

a) *Static cracking*— For $\kappa < \kappa^*$, cracks propagate slowly. Correlations in positions and orientations are weak between successive cracks. Defining $D(c)$ as the probability density for the number of cracks per event, c, the cumulative distribution $D_>(c) = \int_c^{2L^2} \mathrm{d}c'\, D(c')$ decays exponentially; it is short-range (see fig. 2 *a*)). The spatial distribution of fragments is homogeneous, as shown in fig. 1.

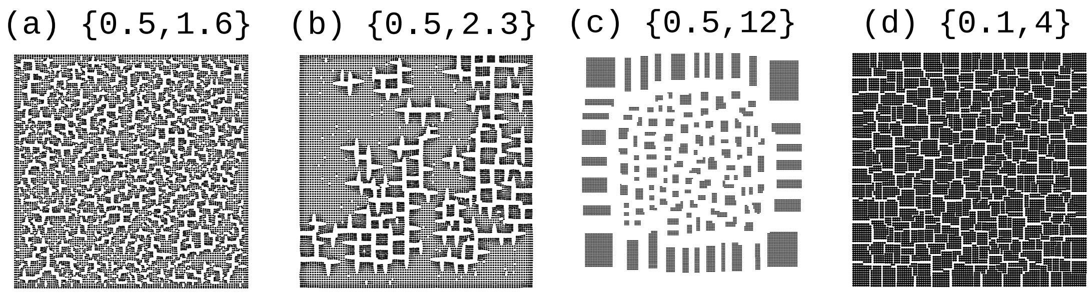

Fig. 1. – Typical patterns for $L = 100$. Legends stand for $\{s, \kappa\}$; *a*) below transition, shows localized cracks and small fragments; *b*) near transition, almost percolating cracks and larger fragments; *c*) inhomogeneity well above transition; *d*) in the one-phase region, shows more isotropic cracks.

b) *Dynamic cracking*— For $\kappa > \kappa^*$, cracks percolate the sample and break it in a catastrophe. $D_>(c)$ is long-range with a long plateau and a sharp roll-off (see fig. 2 *a*)), corresponding to a peak there in $D(c)$. The cut-off $c_{\max}$ scales as L^2. The distribution of fragments is inhomogeneous deep in this region, as shown in fig. 1 *c*). Along the phase boundary at $\kappa = \kappa^*(s)$, $D_>(c) \sim c^{-\eta}$ (see fig. 2 *b*)) with an exponent $\eta \approx 0.75 \pm 0.05$ to 0.63 ± 0.02 as s varies from 0.8 to 0.5 [13].

By probing the κ and L dependence of $c_{\max}$, we determine the phase diagram, fig. 2 *c*). The transition between static and dynamic cracking becomes less striking as s is decreased. Beyond the end point (s_c, κ_c) [14], the two phases become indistinguishable. Traversing along fixed s within the one-phase region, we find in $c_{\max}$ *vs.* κ a mild peak which saturates for large L, and $D_>(c)$ eventually decays exponentially at large c, indicating non-percolating cracks.

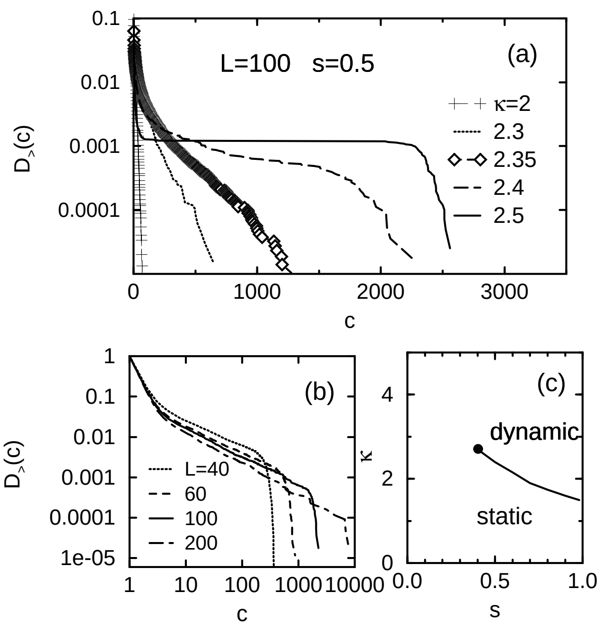

Fig. 2. – *a*) Cumulative distribution of crack size per event. Note short- (long-) range correlation below (above) $\kappa^* \approx 2.4$; *b*) power law decay $D_>(c) \sim c^{-0.63}$ at $\kappa = \kappa^*$ and $s = 0.5$; *c*) the phase diagram with a critical end point.

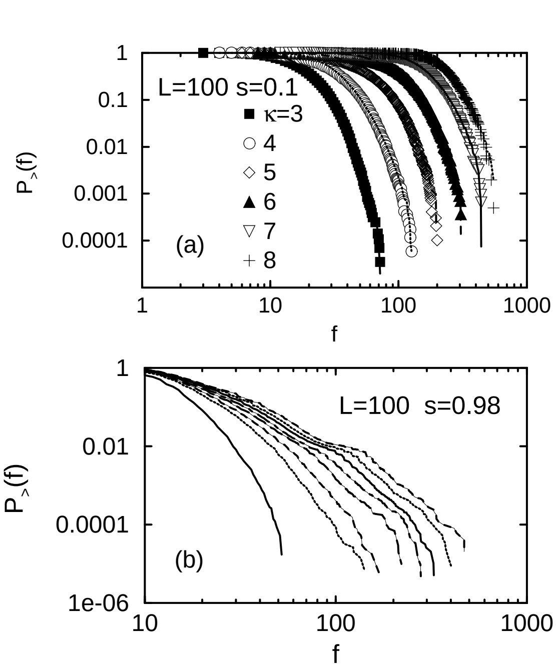

Fig. 3

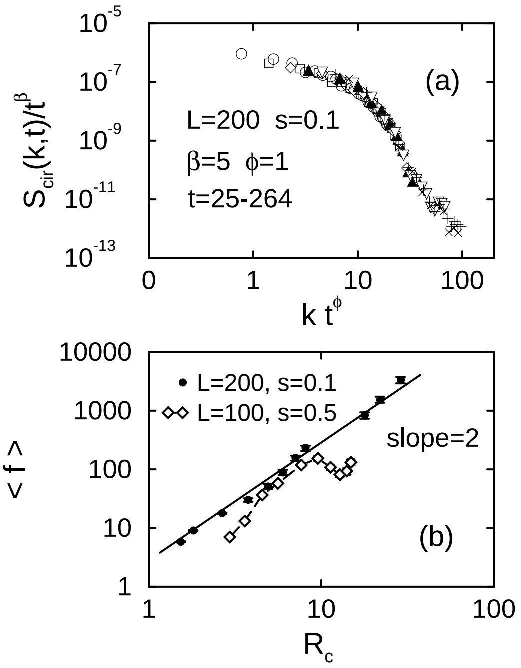

Fig. 4

Fig. 3. – Cumulative distribution of fragment area. *a*) shows good fits (lines) to log-normal distribution for non-percolating cracking; and *b*) shows deviation from log-normal and emergence of a kink (spatial inhomogeneity) for dynamic cracking, where $\kappa = 2$, 3, 3.5, 4, 4.5, 5, 5.5, and 6 from left to right.

Fig. 4. – *a*) Scaling of structure factor for stress field prior to cracking, corresponding to $R(t) = 30$–180. *b*) Mean fragment area scales with correlation length of stress field for static cracking (upper curve), but not for dynamic cracking (high end of lower curve).

Also of particular interest is the probability density $P(f)$ of fragment area f, the center of focus of fragmentation theories [15], [16] and experiments [17]. For non-percolating cracking, $P(f)$ is best described by a log-normal distribution [18]

$$P(f) = \frac{1}{\sqrt{2\pi\sigma^2} f} \exp\left[\frac{-(\ln f - \mu)^2}{2\sigma^2}\right], \tag{2}$$

as shown in fig. 3 *a*) for the cumulative distribution $P_>(f) = \int_f^{L^2} \mathrm{d}f'\, P(f')$. The same form is found for κ at and near κ^*. On the other hand, $P_>(f)$ for dynamic cracking deviates from log-normal for significantly larger κ: a kink gradually appears and becomes more pronounced (see fig. 3 *b*)), implying different statistics for the fragments near the boundary and the center. Although the tails of $P_>(f)$ suggest a power law dependence, we cannot be certain due to the limited range of data.

A consistent physical picture emerges from studying the reduced stress field $\varepsilon(\mathbf{r}, t) \equiv |\mathbf{b}|/F_\mathrm{c} \propto \sqrt{\mathrm{Tr}\boldsymbol{\sigma}^2}/F_\mathrm{c}$, where $\boldsymbol{\sigma}$ is the local stress tensor, and $0 < \varepsilon < 1$. We compute the structure factor $S(\mathbf{k}, t) = \langle|\tilde{\varepsilon}(\mathbf{k}, t)|^2\rangle/L^2 - L^2\delta_{\mathbf{k},0}\langle\overline{\varepsilon(\mathbf{r}, t)}\rangle^2$, where $\tilde{\varepsilon}$ is the Fourier transform, and the overline means a spatial average. Averaging over the circular average $S_\mathrm{cir}(k, t)$, we define a characteristic length of the stress field $R(t) = \langle k^2\rangle_\mathrm{cir}^{-1/2}$. Before cracking occurs, block slips start from the edges and propagate inward. Thus, ε is reduced near the edges but enhanced linearly in time in the bulk. These changes are reflected in the growth of $R(t)$ via a power law (cf. [9]) $R(t) \sim t^{\phi}$. Within statistical uncertainty, $\phi \approx 1$ for all s asymptotically at large L. The self-similar build-up of correlation is evident in the temporal scaling for $t < t_c$: $S_{\rm cir}(k,t) = t^{\beta}\Phi(kt^{\phi})$ (see fig. 4 *a*)). Hence, $R(t)$ may be interpreted either as the penetration depth of slip events into the bulk or the correlation length of the stress field, whose growth towards L is a manifestation of the approach to a critical state self-organized by the stick-slip mechanism [10], [11].

The time t_c when cracking sets in may be deduced by subjecting the enhanced stress $|\mathbf{b}| \approx Ks$ in the bulk to the threshold condition $\varepsilon \approx Ks/F_c = ts/\kappa \geq 1$. Thus $t_c \approx \kappa/s$, which has been verified. Whether the initial crack triggers an instability depends on the strength of the stress $\mathcal{S}$ at a crack tip relative to the spatial fluctuation $\delta\varepsilon$. Since $\delta b \sim KA$, we have $\delta\varepsilon/\varepsilon \propto A/s$. So, for small s, $\mathcal{S}$ is suppressed relative to $\delta\varepsilon$, fluctuations dominate and the cracks are localized. This explains the absence of percolating cracks in the small-s regime.

For large s, the transition can be understood by virtue of the effect of κ on $\mathcal{S}$. $\mathcal{S}$ is large only if there are stress transfers from crack sides to tips by slipping those blocks that link to broken springs. The ratio $F/F_s \sim Ks \approx \kappa$ determines if they slip, where $F \sim Ks$ in the bulk (see eq. (1)) and $K \approx t_c$ have been used. Since κ has little influence on $\delta\varepsilon$, we conclude that small κ gives rise to small $\mathcal{S}$ and hence isolated cracks. The cracks bordering a typical fragment are then highly localized and weakly correlated, the usual argument leading to a log-normal distribution applies [18], [15]. On the other hand, large κ fulfills the slipping criterion easily, we have substantial stress transfers at a crack opening and hence a large $\mathcal{S}$, which forces more broken springs. An instability is inevitable. The strong correlations of cracks (as shown in $D_{>}(c)$) invalidates the assumption for log-normal distribution.

Further insight can be gained from exploiting $R(t)$. For non-percolating cracking, we find the mean fragment area $\langle f\rangle \sim R_c^2$, where $R_c \equiv R(t=t_c)$ (see fig. 4 *b*)). This implies $R(t)^2$ *prior to* cracking sets a lower bound of $\langle f\rangle$ for the *final* cracked state. Using $t_c \approx \kappa/s$, we find $\langle f\rangle \propto \kappa^{2\phi}$. This simple relation highlights the crucial role of κ in our model. In contrast, for dynamic cracking, the memory of R_c and hence this predictability are lost as ε is extensively modified by the block slips at crack openings.

Finally, our model is robust in the following sense. If we allow F_c and F_s to increase with K, the simplest alternatives are: a) $F_c \propto K$, $F_s = \text{const}$, so $\kappa \propto K$; b) $F_s \propto K$, $F_c = \text{const}$, so $\kappa \propto 1/K$; and c) $F_s \propto F_c \propto K$, $\kappa = \text{const}$. In the first case, the final state is entirely controlled by F_c/K. It undergoes a first-order transition from a crack-free to a cracked state as $F_c/K \to s + 2A \equiv \zeta$ from above. Further below ζ, the density of broken springs quickly saturates. The second case is trivial, having all the springs broken at long times. For the third case, either nothing happens if neither thresholds are exceeded, or the updating sequence is ambiguous if both are. Thus, within our formalism, our choice of rules is the only one that evolves into a non-trivial cracked state through the interplay among sticking, slipping and cracking, without the need of fine tuning the parameters or initial conditions.

Conclusion. – We have introduced a robust spring-block model of surface fracture. Despite its simplicity, it captures the correct trends observed in experiments and displays rather complex behavior, which can be understood from the evolution of the stress field and characterized by dimensionless control parameters. Concerning the fragment distributions, similar transitions between log-normal and power law distribution have been observed in some recent fragmentation experiments [17], similar to ours, a substrate could be utterly different.

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[12] The amplitude A of the initial random displacement (x, y) is an additional parameter. We chose $A = 0.02$ and found the same qualitative features for $A = 0.002$–0.2.

[13] Other exponents and the order parameter may be defined in similar ways as for percolation, see, *e.g.*, Stauffer D. and Aharony A. *Introduction to Percolation Theory*, 2nd edition, (Taylor and Francis, London) 1992.

[14] Larger A has more disorder, giving a shorter and higher phase boundary with a greater s_c.

[15] See, *e.g.*, Redner S., in [1], and references therein.

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