

## Generalization of a Two-Dimensional Burridge-Knopoff Model of Earthquakes

Kwan-tai Leung <sup>(1)</sup>, Judith Müller <sup>(2)</sup> and Jørgen Vitting Andersen <sup>(2,\*)</sup>

<sup>(1)</sup> Institute of Physics, Academia Sinica, Nankang, Taipei 11529 Taiwan, ROC

<sup>(2)</sup> Department of Physics, McGill University, Rutherford Building, 3600 University Street, Montréal, Québec, Canada H3A 2T8

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**Abstract.** — We present a generalization of the two-dimensional spring-block model of earthquakes previously studied by Olami, Feder and Christensen (*Phys. Rev. Lett.* **68** (1992) 1244). Making the simplest possible assumption, we regard the tectonic plates as elastic media with inter- and intra-plate harmonic forces, that is, forces governed by Hooke's law. The robustness of the model with respect to effects of internal strain, vectorial force and different boundary conditions are examined and demonstrated both analytically and numerically.

In 1967, Burridge and Knopoff [1] introduced a one-dimensional (1D) system of springs and blocks to study the role of friction along a fault in earthquakes. Since then, many other researchers have investigated similar dynamical models of many-body systems with friction, ranging from propagation and rupture in earthquakes [2-11] to the fracture of overlayers on a rough substrate [12].

Among these developments, a deterministic version of the 1D Burridge-Knopoff (BK) model was analyzed by Carlson and Langer [2] and the same model but in a quasi-static limit was studied by Nakanishi [3]. A 2D quasi-static variant was first simulated by Otsuka [4] and later by Brown, Scholz and Rundle [5], who formulated it as a discrete cellular automaton. A similar model with non-conservative, continuous local variable (the force), generalizing the model of Bak, Tang and Wiesenfeld [6], was first introduced by Feder and Feder [7] in connection with their experiment. A more refined version was later developed by Olami, Feder and Christensen [8] (OFC). Contrary to previous models, the model by OFC (henceforth called the OFC model) is derivable under certain limit (see below) from a 2D BK model, thereby establishing a more direct connection between earthquake problems and self-organized criticality. One of its interesting features is the dependence of the values of critical exponents on the level of conservation [8, 10]. OFC argued that it explains the variance of the exponent in the Gutenberg-Richter [13] law observed in real earthquakes [14]. More recently, motivated by the findings of OFC, János and Kertész [15] and Middleton and Tang [16] addressed the important issue of how a model without conservation law may attain self-organized criticality.

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(\*) Author for correspondence (e-mail: j.vitting@ic.ac.uk)

*Present address:* Department of Mathematics, Imperial College, Huxley Building, 180 Queen's Gate, London SW7 2BZ, GB

Making the simplest possible assumption, we regard the tectonic plates as elastic media with inter- and intra-plate harmonic forces, that is, forces governed by Hooke's law. It is worth noting that the OFC model corresponds, in the sense of the specification of forcings, to a restricted form of a 2D BK model. In this article, we shall examine the effects on the physical properties of the OFC model when the restrictions are removed. The intense interests centered on the OFC model and nonconservative models in general motivate such an examination. We shall also discuss the influence and physical realization of certain boundary conditions.

## Model

As before, our model consists of a 2D array of blocks in contact with a rough surface. Each block is interconnected to its nearest neighbors *via coil* springs whose spring constants are  $K_1$  and  $K_2$  and unstretched lengths  $l_1$  and  $l_2$ , along the  $x$ - and  $y$ -direction respectively. Each block is also connected to a rigid driving plate by a *coil* spring of spring constant  $K_L$ . The purpose of the coil spring is to confine the block within the  $x$ - $y$  plane. The position of the coil springs on the moving plate, labeled by  $(i, j)$  where  $1 \leq i, j \leq L$ , forms a square lattice with lattice constants  $a_1, a_2$ . We restrict ourselves to the situation where  $a > l$ , and the displacements  $x_{i,j}, y_{i,j}$  measured from  $(i, j)$  fulfill  $x_{i,j} \ll a_1$  and  $y_{i,j} \ll a_2$ , so that Hooke's law applies. The plate moves at constant, infinitesimal speed. Stress thereby builds up between the array and the plate. The friction from the underlying rough surface prevents a block from moving, until it exceeds a static threshold  $F_{th}$  and the block then slips instantaneously to a new equilibrium position.

While this setup is an obvious extension of the 1D BK model [1] to two dimensions, previous models [4, 5, 8] invariably correspond to a different setup in which the blocks are confined to move in one direction. To our knowledge, this restriction was not physically motivated, but introduced for the reason of simplicity. It is not obvious *a priori* whether the physical properties of the model will be affected significantly. To find out, we lift this restriction and start with the equations for the force appropriate to coil spring connections. The net force  $\mathbf{F}_{i,j}$  (which equals the friction from the rough surface) on a block at  $(i, j)$  is now a vector:

$$\mathbf{F}_{i,j} = \mathbf{f}_L + \mathbf{f}_{(i+1,j)-(i,j)} + \mathbf{f}_{(i-1,j)-(i,j)} + \mathbf{f}_{(i,j+1)-(i,j)} + \mathbf{f}_{(i,j-1)-(i,j)} \quad (1)$$

where  $\mathbf{f}_{(i\pm 1,j\pm 1)-(i,j)}$  is the force exerted by a neighboring block and  $\mathbf{f}_L$  the loading force by the driving plate. Specifically, we use Hooke's law:

$$f_{(i+1,j)-(i,j)}^x = K_1 \left[ a_1 + x_{i+1,j} - x_{i,j} - \frac{(a_1 + x_{i+1,j} - x_{i,j})l_1}{\sqrt{(a_1 + x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}} \right], \quad (2)$$

$$f_{(i+1,j)-(i,j)}^y = K_2 \left[ y_{i+1,j} - y_{i,j} - \frac{(y_{i+1,j} - y_{i,j})l_2}{\sqrt{(a_1 + x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}} \right]. \quad (3)$$

Then the  $x$  component of the force in equation (1) takes the form:

$$\begin{aligned} F_{i,j}^x &= f_L^x - K_1 [2x_{i,j} - x_{i+1,j} - x_{i-1,j} \\ &+ \frac{(a_1 + x_{i+1,j} - x_{i,j})l_1}{\sqrt{(a_1 + x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}} - \frac{(a_1 - x_{i-1,j} + x_{i,j})l_1}{\sqrt{(a_1 - x_{i-1,j} + x_{i,j})^2 + (y_{i,j} - y_{i-1,j})^2}}] \\ &- K_2 [2x_{i,j} - x_{i,j+1} - x_{i,j-1} \\ &+ \frac{(x_{i,j+1} - x_{i,j})l_2}{\sqrt{(a_2 + y_{i,j+1} - y_{i,j})^2 + (x_{i,j+1} - x_{i,j})^2}} + \frac{(x_{i,j-1} - x_{i,j})l_2}{\sqrt{(a_2 - y_{i,j-1} + y_{i,j})^2 + (x_{i,j} - x_{i,j-1})^2}}] \end{aligned} \quad (4)$$

and, by symmetry,  $F_{i,j}^y$  follows by switching  $x \leftrightarrow y$ ,  $i \leftrightarrow j$ ,  $l_1 \leftrightarrow l_2$ ,  $a_1 \leftrightarrow a_2$  and  $K_1 \leftrightarrow K_2$ .

### Linear Version

To compare with the OFC model, we expand  $\mathbf{F}_{i,j}$  to first order in the displacements  $x$ 's and  $y$ 's. Specifying  $\mathbf{f}_L = (-K_L x_{i,j}, 0)$  as in [5, 8], we readily find that

$$\begin{aligned} F_{i,j}^x &= -K_L x_{i,j} - K_1(2x_{i,j} - x_{i+1,j} - x_{i-1,j}) - K_2 S_2(2x_{i,j} - x_{i,j+1} - x_{i,j-1}) + \dots \\ F_{i,j}^y &= -K_1 S_1(2y_{i,j} - y_{i+1,j} - y_{i-1,j}) - K_2(2y_{i,j} - y_{i,j+1} - y_{i,j-1}) + \dots \end{aligned} \quad (5)$$

Thus a slip to a zero-force position for the block at  $(i, j)$  results in the following force redistribution [17, 18]:

$$\begin{aligned} F_{i\pm 1,j}^x &\rightarrow F_{i\pm 1,j}^x + \alpha_1 F_{i,j}^x, \\ F_{i,j\pm 1}^x &\rightarrow F_{i,j\pm 1}^x + S_2 \sigma \alpha_1 F_{i,j}^x, \\ F_{i\pm 1,j}^y &\rightarrow F_{i\pm 1,j}^y + \frac{S_1}{\sigma} \alpha_2 F_{i,j}^y, \\ F_{i,j\pm 1}^y &\rightarrow F_{i,j\pm 1}^y + \alpha_2 F_{i,j}^y, \\ \mathbf{F}_{i,j} &\rightarrow 0, \end{aligned} \quad (6)$$

where  $S_1 \equiv (a_1 - l_1)/a_1$  is the internal strain of the network in the  $x$ -direction, and similarly for  $S_2$ .  $\sigma \equiv K_2/K_1$  and  $\kappa \equiv K_L/K_1$  are measures of anisotropies in the couplings. In the bulk,  $\alpha_1$  and  $\alpha_2$  are given by:

$$\alpha_1 = \frac{1}{2(1 + S_2 \sigma) + \kappa}; \quad \alpha_2 = \frac{1}{2(1 + S_1/\sigma)}. \quad (7)$$

Using equation (6), our model can be described as a coupled map lattice as was done in [8]. If  $y_{i,j} \equiv 0$  is imposed at all sites as in [5, 8],  $F_{i,j}^y = 0$  to first order and we recover the OFC model in the limit  $S_2 \rightarrow 1$ , *i.e.*, for *maximal* internal strain along the  $y$ -direction. This also follows from the general relation equation (4), since  $F^x$  becomes linear in  $x$  for  $S_2 = 1$  and  $y = 0$ . The physical rationale behind this correspondence is that *leaf* springs were chosen along  $y$  in the array in references [4, 5, 8], which by definition can only be *bent but not extended*. To linear order, they are effectively fully stretched coil springs. The advantage of having different kinds of springs along  $x$ - and  $y$ -direction is that the force is scalar and the equations linear. However, the network is intrinsically asymmetric (not to be confused with anisotropy). This manifests itself most clearly in the elastic moduli [20]: for the OFC model, we obtain [17]  $C_{1111} = K_1 a_1/a_2$ ,  $C_{2222} = \infty$ ,  $C_{1122} = 0$ , and  $C_{1212}$  (shear modulus)  $= K_2 a_2/a_1$ . Clearly, setting  $K_1 \equiv K_2$  does not render the model symmetric. In contrast, our model is symmetric:  $C_{1111} = K_1 a_1/a_2$ ,  $C_{2222} = K_2 a_2/a_1$ ,  $C_{1122} = 0$ , and  $C_{1212} = (K_1 K_2 S_1 S_2 a_1 a_2)/(K_1 S_1 a_1^2 + K_2 S_2 a_2^2)$ .

Before going further, we remark that given the underlying square lattice, neither model is isotropic. More importantly, neither satisfies a "space filling" condition in that they do not contract laterally when stretched. This is obvious from the way the springs are connected, and is also reflected in the above relation  $C_{1122} = 0$ , which gives a zero Poisson's ratio. This is a shortcoming of most if not all of the spring-block models. However, the space filling condition may be fulfilled if extra springs between *next*-nearest neighbors are added whose spring constant will then be proportional to  $C_{1122}$  [17]. Since the addition of extra springs introduces considerable complications into the rules of the cellular automata, we will not pursue it here. Notwithstanding such complications associated with the implementation, it is useful to remember that the discrete spring-block models may be systematically refined along this line, and the general, tensorial macroscopic elastic equations may be obtained in the continuum limit.

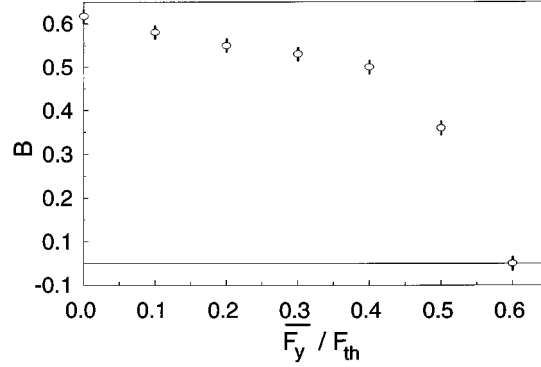


Fig. 1. — Exponent in slip-size distribution  $P(> s) \sim s^{-B}$  versus conservation level of a force component, from data of  $L = 100$ ,  $S_1 = S_2 = 1$ , and  $\kappa = K_1 = K_2 = F_{th} = 1$ , with FBC.

For general strain ( $0 \leq S_2 < 1$ ) but  $F^y \equiv 0$ , the linear version of our model coincides with the so called “anisotropic” OFC model [8] with our  $S_2\sigma$  corresponding to OFC’s  $K_2/K_1$ . The statement [8] regarding the variance in the Gutenberg-Richter law as a result of different coupling strengths has to be reinterpreted, in the present context, as also a result of different internal strain.

Next, for the more general case of  $y_{i,j} \neq 0$ , the two force components are locally coupled because a slip is decided by the sign of  $\sqrt{(F^x)^2 + (F^y)^2} - F_{th}$ . We need to consider its relevance. We examine the *net* change of force after a block *in the bulk* at  $(i, j)$  slips by a distance  $(\delta x, \delta y)$ :

$$\delta F^x = -K_L \delta x = -\kappa \alpha_1 F_{i,j}^x; \quad \delta F^y = 0. \quad (8)$$

This follows from equations (6, 7), showing that  $F^y$  is conserved in the bulk. The changes at the boundary depend on the boundary conditions.

Following OFC, for “free” boundary conditions (FBC) there are three neighbors along the edges and two at corners. Thus, the  $\alpha$ ’s on the boundary differ from the bulk values in equation (7):

$$\begin{aligned} \alpha_1^{(x=1,L)} &= \frac{1}{1 + 2S_2\sigma + \kappa}, & \alpha_1^{(y=1,L)} &= \frac{1}{2 + S_2\sigma + \kappa}, & \alpha_1^{(corner)} &= \frac{1}{1 + S_2\sigma + \kappa}; \\ \alpha_2^{(x=1,L)} &= \frac{1}{2 + S_1/\sigma}, & \alpha_2^{(y=1,L)} &= \frac{1}{1 + 2S_1/\sigma}, & \alpha_2^{(corner)} &= \frac{1}{1 + S_1/\sigma}. \end{aligned} \quad (9)$$

As a result, equation (8) holds at all sites. Although equation (8) implies  $|F^x|$  always decreases in a slip event, the spatial mean  $\bar{F}^x$  fluctuates about a finite value due to constant loading. But due to isolation for FBC, the spatial mean  $\bar{F}^y$  is identically zero by Newton’s third law. In the steady state, we find that the fluctuation of  $F^y$  also approaches zero, so that this coupling of a conservative component  $F^y$  to  $F^x$  is irrelevant for FBC.

Although  $\bar{F}^y = 0$  in practice, it remains interesting to see, generically, if a coupling to a finite conservative field  $\bar{F}^y = \text{const} \neq 0$  [19] is relevant in the context of more general cellular automata, analogous to similar consideration in critical dynamics [21]. Figure 1 shows that  $\bar{F}^y$  matters, as the exponent  $B$  defined in the Gutenberg-Richter law varies continuously. The OFC model with FBC corresponds to the point  $F^y \equiv 0$ .

For “open” boundary conditions (OBC) there are four neighbors throughout. Equation (7) then holds at all sites but equation (8) is modified to  $\delta F^y/F^y < 0$  at the boundary.

Table I. — Comparisons between the OFC model and our generalizations for their critical behavior.  $F_{i,j}^y$  denotes the  $y$ -component of the force acting on a block, and  $\bar{F}^y$  is its spatial average.

Boundary conditions	Specifications of $F^y$	Critical behavior <i>cf.</i> OFC model
FBC	$F_{i,j}^y \equiv 0$	same
	$F_{i,j}^y \neq 0, \bar{F}^y = 0$	same
OBC	$F_{i,j}^y \neq 0, \bar{F}^y \neq 0$	different, new universality classes
	$\bar{F}^y \neq 0$	same (stable fixed point at $\bar{F}^y = 0$ )

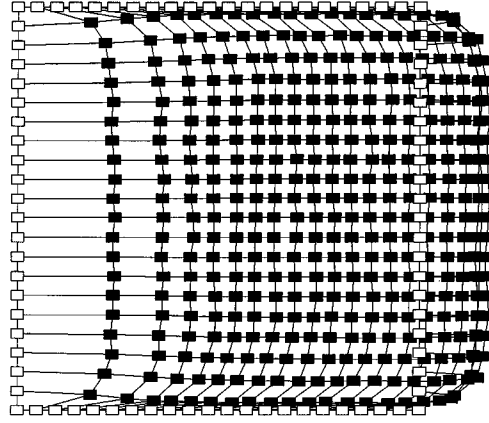


Fig. 2. — Snapshot of a configuration of blocks after 2000 avalanches, for  $L = 20$ ,  $S_1 = S_2 = 0.9$ , and  $\kappa = K_1 = K_2 = F_{th} = 1$ , showing the unphysical effect of pinned frame (denoted by  $\square$ ) with OBC. The array is pulled to the right. The system evolves according to equation (10).

This implies that a system with arbitrary initial spatial mean,  $\bar{F}^y$ , will always flow *in steady state* to the stable fixed point at  $\bar{F}^y = 0$ .

Therefore, we have shown that the OFC model is stable against vectorial perturbation for both the free and open boundary conditions studied by OFC. Our results are summarized in Table I. The physical origin of these results can be traced to the manner of loading along a fixed direction (*cf.* Eq. (8)).

Since the boundary conditions are always important ingredients of the model, we digress for a moment to discuss the physical realization of the OBC, which is defined [8] by an “imaginary layer of blocks” connected around the system in order to have the same number of nearest neighbors everywhere. Physically, such a layer corresponds to a rigid frame that attaches to the array by springs. Since spatially uniform loading in reference [8] implies no relative displacement between the frame and the array during loading, the imaginary layer never slips: it is permanently pinned on the rough surface. Although this is an unphysical setup, it is far from obvious in the  $F_{i,j}$  representation. Working with the linearly related  $x_{i,j}$  instead, we show in Figure 2 how the initially square array of blocks is distorted due to the pinning. At long times, the distortion can be arbitrarily large so that there is no meaningful steady state for this model. To make physical sense, the frame has to move along with the driving plate. It is then intuitively

clear that loading cannot be uniform: due to pulling and pushing by the frame, boundary blocks are loaded more than in the bulk (*e.g.*,  $\Delta F_{1,j}^x = (1 + \kappa^{-1})\Delta F_{\text{bulk}}^x$ ). Incorporating such non-uniformity, we observe numerically [17] that the model becomes noncritical and reaches periodic states much like with periodic boundary conditions [10]. Therefore, we conclude that the OBC is a mathematical convenience in that the model is rendered noncritical if it is implemented in a physical way.

### Nonlinear Version

Finally, equation (4) allows for an investigation of nonlinear effects which naturally arise from the spring connections. However, it is no longer possible to formulate the model as a coupled map lattice (*cf.* Eq. (6)), because the displacements have to be kept track of. For each slip event, the equations  $\mathbf{F} = 0$  have to be solved for the displacements  $(\tilde{x}, \tilde{y})$  that defines the zero-force position of the block.

Apparently this program is not computationally efficient for large avalanches and high orders. But the second order is simple: all second order terms ( $\tilde{x}^2, \tilde{y}^2, \tilde{x}\tilde{y}$ ) cancel *in the bulk* (*i.e.*, when  $(i, j)$  has four nearest neighbors), resulting in two coupled *first* order equations for  $(\tilde{x}, \tilde{y})$  in terms of the  $\mathbf{F}$  and  $(x, y)$  before the slip:

$$\begin{aligned} \frac{F_{i,j}^x}{K_1} &= (\tilde{x}_{i,j} - x_{i,j})[\alpha_1^{-1} + \frac{\sigma l_2}{a_2^2}(y_{i,j+1} - y_{i,j-1})] \\ &+ (\tilde{y}_{i,j} - y_{i,j})[\frac{l_1}{a_1^2}(y_{i+1,j} - y_{i-1,j}) + \frac{\sigma l_2}{a_2^2}(x_{i,j+1} - x_{i,j-1})], \end{aligned} \quad (10)$$

plus a corresponding equation for  $F_{i,j}^y$ . It yields nonlinear dependence of  $(\tilde{x}, \tilde{y})$  on  $(x, y)$  *via* the terms proportional to  $l/a^2$ . Due to missing neighbors of the blocks on the edges with FBC, second order terms in  $(\tilde{x}, \tilde{y})$  survive and one needs to solve higher order (3rd and 4th) equations for the equilibrium positions of the boundary blocks.

Based on equation (10), we have performed simulations for different boundary conditions. Remarkably, even with substantial nonlinearities (measured by  $l/a^2$ ), we do not find any deviation from the linear behavior for FBC, OBC and PBC. Thus, we are inclined to believe that the important nonlinearities are not associated with the spring actions, but rather with the force redistributions during the stick-slip motion, which are the same in the linear and nonlinear cases.

### Summary

Using Hookean force-displacement relations, we have investigated the critical behavior of a 2D spring-block model of earthquakes under the quasi-static limit. We find that the internal strain is an additional ingredient in the variance of the Gutenberg-Richter law. As a consequence of loading along a fixed direction, the model studied by Olami *et al.* [8] displays striking stability against generalization to vectorial force for both free and open boundary conditions. However, if the loading is applied in both directions (*e.g.*,  $f_L = -(K_L^x x, K_L^y y)$ , or as shears through the boundary), the vectorial generalization is expected to be necessary. Finally, nonlinear effects associated with the spring actions are found to be not important.

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- [18] In 2D, the equations of motion in the continuum approach of references [2, 11] to first order in the displacements  $\mathbf{U} \equiv (U_x, U_y)$ , including strain  $S$ , are similarly modified:

$$\begin{aligned}\partial_t^2 U_x &= \xi^2 (\partial_x^2 U_x + S \partial_y^2 U_x) - U_x - \partial_t U_x \phi(|\partial_t \mathbf{U}|) \\ \partial_t^2 U_y &= \xi^2 (S \partial_x^2 U_y + \partial_y^2 U_y) - \partial_t U_y \phi(|\partial_t \mathbf{U}|),\end{aligned}$$

where  $\phi(|\partial_t \mathbf{U}|)$  is a speed-dependent friction term and  $\xi$  a characteristic length. The equation (10) of reference [11] may be obtained by setting  $S = 1$  and  $U_y = 0$ .

- [19] A physical realization of  $\bar{F}^y = \text{const} \neq 0$  would correspond to a case where the spring-block system is moving in the  $x$ -direction but being pushed by an external agent (*e.g.* due to a transverse tectonic plate movement) in the  $y$ -direction.
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