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Self-organized criticality in an isotropically driven model approaching equilibrium

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Abstract

Self-organized criticality (SOC) in systems governed by stick-slip dynamics is studied by means of a spring-block model. Contrary to conventional models which are loaded additively in their stationary states, our model is driven towards equilibrium in a multiplicative and isotropic manner via a temporally increasing coupling strength. A coarsening process builds up correlations in the bulk and establishes power-law avalanche size distributions, independent of boundary conditions and system parameters. The observed behavior is robust and suggests that some experimental systems approaching equilibrium via a punctuated threshold dynamics may well show SOC.

1. Introduction

Stick-slip mechanism plays a dominant role in a wide variety of physical phenomena, ranging from earthquakes [1], rupture [2], granular flows [3] to interface pinning [4,5]. In particular, it is often the means of introducing dissipation into many model systems exhibiting self-organized criticality (SOC) [6].

SOC refers to the spontaneous organization in dissipative systems towards the kind of dynamical critical state which is characterized by power laws. Of considerable recent interests is a class of SOC models known as the spring-block models. Such models are coarse-grained descriptions of systems whose elements are in frictional contact with a substrate, where stick-slip action arises from external driving of the elements. An interesting example is the one proposed by Olami, Feder and Christensen (OFC) [7],

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much studied because of its coexistence of SOC and non-conservation [7–11]. It is now understood qualitatively that it belongs to a class of models exhibiting SOC by means of de-synchronization which is initiated and invaded into the bulk from inhomogeneities such as open boundaries or quenched disorders [11]. The de-synchronized state has long-range correlation and power-law distributed avalanche size, thus qualifying an SOC state. In contrast, the pure model without inhomogeneity is known to settle into a periodic state with only size-one avalanches [9–11]. Although temporally more organized in the avalanches where each site takes turn to topple, it has short-range spatial correlation in the sense that the topplings are localized. In particular, the model with periodic boundary conditions (PBC) is not critical. PBC is also known to forbid criticality from occurring in conservative systems such as sandpile models [6,12].

Here we introduce a new class of spring-block models which differ from previous stick-slip models in several aspects [13]:

1. It is driven *multiplicatively and isotropically* via a temporally increasing coupling strength [14,15]. Previous models are driven additively, realized for example by adding grains on a pile one at a time [6,12], or by pulling the blocks along a fixed direction [16–19,7–11].
2. It achieves SOC by a novel mechanism without invoking any spatial inhomogeneity. In particular, it is insensitive to boundary conditions, hence critical even under PBC.
3. It exhibits the dual property of *approaching* (without ever reaching) an equilibrium state and being characterized by an effective steady state in the dynamical variables. In contrast, previous SOC models settle into nonequilibrium steady states.

To our knowledge, our model is the first of its kind to show SOC in approach to an equilibrium state, thereby suggesting a new class of experimental systems in search of real SOC systems.

2. An isotropically driven model

Similar to the OFC model [7], we consider a two-dimensional array of blocks interconnected among nearest neighbors by coil springs. The springs are characterized by spring coefficient K and relaxed spring length l. The blocks are in contact with a rough surface, with a static threshold F_s for friction. The array is initially stretched and maintained by friction at a lattice spacing $a > l$, i.e., it is prepared with an internal strain given by $s = 1 - l/a > 0$. Initial spatial disorder is introduced by placing each block at $(ia + x, ja + y)$ with random amplitudes $|x| \leqslant A$, $|y| \leqslant A$. Each block is labeled by the pair of indices $i, j = 1, \ldots, L$. Hereafter we set $a = 1$.

To compare with previous models, the Hookean force–displacement relationships are expanded to first order in (x, y) to obtain the force components acting on a block from its nearest neighbors [20]:

$$\begin{aligned} F^x_{i,j} &= (x_{i+1,j} + x_{i-1,j} - 2x_{i,j})K + (x_{i,j+1} + x_{i,j-1} - 2x_{i,j})sK\,, \\ F^y_{i,j} &= (y_{i+1,j} + y_{i-1,j} - 2y_{i,j})sK + (y_{i,j+1} + y_{i,j-1} - 2y_{i,j})K\,. \end{aligned} \tag{1}$$

Notice that the OFC model barring external pulling terms is recovered by setting $s=1$ and $F^y=0$ [20], reflecting the use of leaf springs in that model [7] (and to our knowledge in all previous spring-block models, see e.g. [16–19] and references therein) along the y-direction. On the other hand, setting $s=0$ yields a set of decoupled one-dimensional (1D) chains, criticality then disappears. Thus, we see that a finite internal strain s is essential for 2D collective behavior (such as SOC) to occur.

We drive the system in the following way. Motivated by stiffening caused by desiccation [21–23] (e.g., drying of clay or mud), we increase the spring constant K at an infinitesimal rate while keeping all the other parameters fixed [14,15]. In a real situation, one may have a more complicated combination of time dependent l, K and F_s. However, our simple definition is more general than it appears, actually applying to a wide class of situations in which K/F_s is an increasing function of time, since only this ratio matters in topplings. Beginning with a stable configuration, suppose the force acting on the block at (i,j) from its neighbors is the maximum among all blocks. Then nothing happens until K is increased to $KF_s/|\boldsymbol{F}_{i,j}|$, when the block slips the distance $(\delta x, \delta y)$ to a new position where $\boldsymbol{F}_{i,j}$ becomes zero:

$$\begin{aligned} F^x_{i,j} &\to 0 = F^x_{i,j} - (2+2s)K\delta x\,, \\ F^y_{i,j} &\to 0 = F^y_{i,j} - (2+2s)K\delta y\,. \end{aligned} \tag{2}$$

This changes the forces acting on its neighbors, e.g.,

$$\begin{aligned} F^x_{i\pm1,j} &\to F^x_{i\pm1,j} + K\delta x\,, \\ F^x_{i,j\pm1} &\to F^x_{i,j\pm1} + sK\delta x\,, \end{aligned} \tag{3}$$

leading to the updating rules for a slip entirely expressed in terms of the forces:

$$\begin{aligned} &F^x_{i,j} \to 0, && F^y_{i,j} \to 0\,, \\ &F^x_{i\pm1,j} \to F^x_{i\pm1,j} + \alpha F^x_{i,j}, && F^y_{i\pm1,j} \to F^y_{i\pm1,j} + s\alpha F^y_{i,j}\,, \\ &F^x_{i,j\pm1} \to F^x_{i,j\pm1} + s\alpha F^x_{i,j}, && F^y_{i,j\pm1} \to F^y_{i,j\pm1} + \alpha F^y_{i,j}\,, \end{aligned} \tag{4}$$

where $\alpha = 1/(2+2s)$. Notice that $\sum_{i,j} \boldsymbol{F}_{i,j}$ is unchanged by a slip, hence the *conservative* nature of the dynamics. As in most other SOC models, an infinite separation of time scales is assumed so that K is increased further only after $|\boldsymbol{F}| \equiv F < F_s$ is restored on all blocks. Thus, in terms of a slow time variable t chosen simply as $t \equiv K$, avalanches occur instantaneously. The central question to ask is under what conditions the system can be critical.

3. Simulation results

We have simulated the model for a wide range of values of s and A, and system size L [13]. PBC is used except otherwise stated. As it turns out, boundary conditions are not essential (see Section 4).

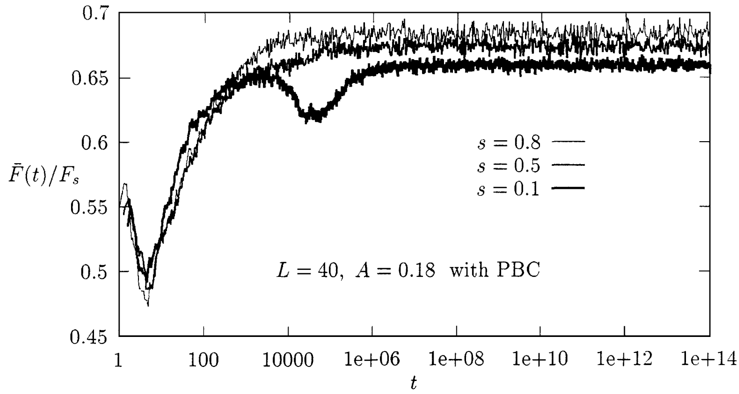

Fig. 1. Spatial averaged force on a block normalized by the threshold vs. time $t \equiv K$, showing approaches to stationarity for different values of internal strain s.

Fig. 1 presents the first evidence that our system reaches an effective stationary state in which the force variable $\bar{F}(t)/F_s$ fluctuates about a finite constant (the overbar denotes a spatial average). Such stationary fluctuations are characteristics of a dynamical steady state in SOC models. Since the ratio measures the effective "distance" of the system from its instability limit, this result implies a marginally stable state in the long time limit, maintained by the threshold condition, preventing the system from ever reaching its equilibrium state where all blocks sit on a perfect regular lattice.

This stationary characteristic is also exhibited in the spatial ordering, measured by the stress field $\sigma(\boldsymbol{r},t)$ that is approximated by the local average of the spring tension. In [15], a characteristic length $R(t)$ was introduced, using the second moment of the radially averaged structure factor $\mathcal{S}(k,t)$ of σ:

$$R(t) \equiv 2\pi \left[\sum_k k^2 \mathcal{S}(k,t) \Big/ \sum_k \mathcal{S}(k,t) \right]^{-1/2} . \tag{5}$$

R measures the correlation of the stress field. Fig. 2 shows how $R(t)$ changes in time for some typical set of parameters. The power-law growth reminds us of a coarsening phenomenon for a concentration field in spinodal decomposition [24]. This analogy is confirmed by directly inspecting the time evolution of $\sigma(\boldsymbol{r},t)$, such as in Fig. 3. Furthermore, the exponent 1/3 holds for *all* $s > 0$ and A, in agreement with the notion of universality and the accepted exponent for conservative dynamics [24].

Notice in Figs. 2 and 3 that R decays to a *stationary* value R_0 after reaching a peak value equal to the system size L. The reason for the decay can be understood as a result of the system remaining in a stressed state at long times, due to the periodic boundary constraints. We remark above, after Eq. (1), on the effect of s on the transverse couplings of the blocks. Such an effect cumulates over a large number of slip events and results in correlation over a distance that is longer for larger s. The value of the stationary correlation also reflects the level of frustration of the system

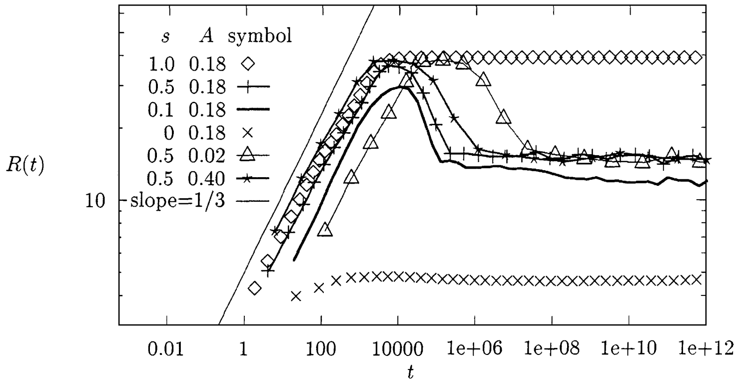

Fig. 2. Characteristic length $R(t)$ of the stress field vs. time for different sets of s and A, showing the universal 1/3 power law and plateau. $L=40$ with PBC.

in lowering its elastic energy: due to PBC, it is unable to shrink its size from L to $(1-s)L$ to achieve a stress-free state. There is more frustration for larger stress s.

An important probe of critical properties is the finite-size effects. Fig. 4 shows that $R(t,L)$ not only scales with L on the plateau (despite $R<L$ for $s<1$), but also satisfies a finite-size scaling form with the asymptotic behavior typical at a critical point:

$$R(t,L)=t^{\phi}\tilde{R}\left(\frac{t}{L^{1/\phi}}\right), \tag{6}$$

where $\phi=0.33\pm 0.01$ is the growth exponent.

Turning our attention to the distributions of event sizes, Fig. 5 shows the change of the distribution of the slip number S as K increases. The phase in which $R\sim t^{1/3}$ corresponds to a gradual departure from an exponential $P(S)$, while the phase with stationary R_0 corresponds to a power-law distribution

$$P(S)\sim S^{-(B+1)}, \qquad B=0.15\pm 0.05\,. \tag{7}$$

From (6) and Fig. 4b, the convergence time is given by $\tau\approx L^{1/\phi}$.

Now from Eq. (2) we have the force drop in one slip given by

$$F^{\mathrm{slip}}=\frac{Ku}{\alpha}, \tag{8}$$

where $u\equiv\sqrt{(\delta x)^2+(\delta y)^2}$ is the slip distance. Since $F^{\mathrm{slip}}\gtrsim F_s$, this implies the slip distance drops like $1/K$ as the positions of the blocks converge to a regular lattice. This raises questions about the observability of $P(S)$ as a result of this approach to equilibrium. While usually only the statistics of S is extracted from avalanches in other SOC models, it is important in our case to look for other stationary distributions of observables.

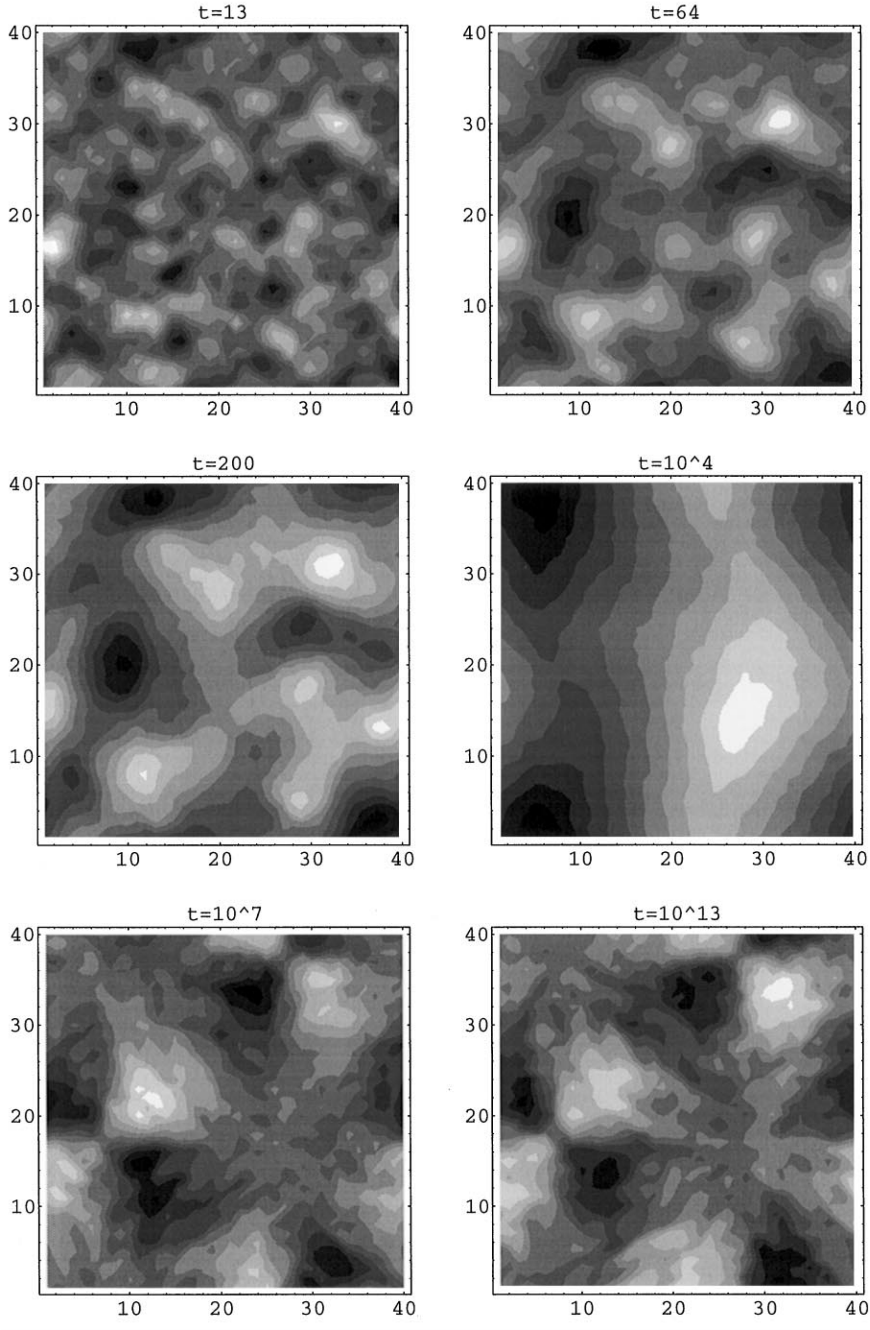

Fig. 3. Evolution of the stress field $\sigma(\boldsymbol{r},t)$ for $L=40$, $s=0.5$, $A=0.18$ with PBC, cf. $R(t;L=40)$ in Fig. 4a. The same time intervals are used in Fig. 5.

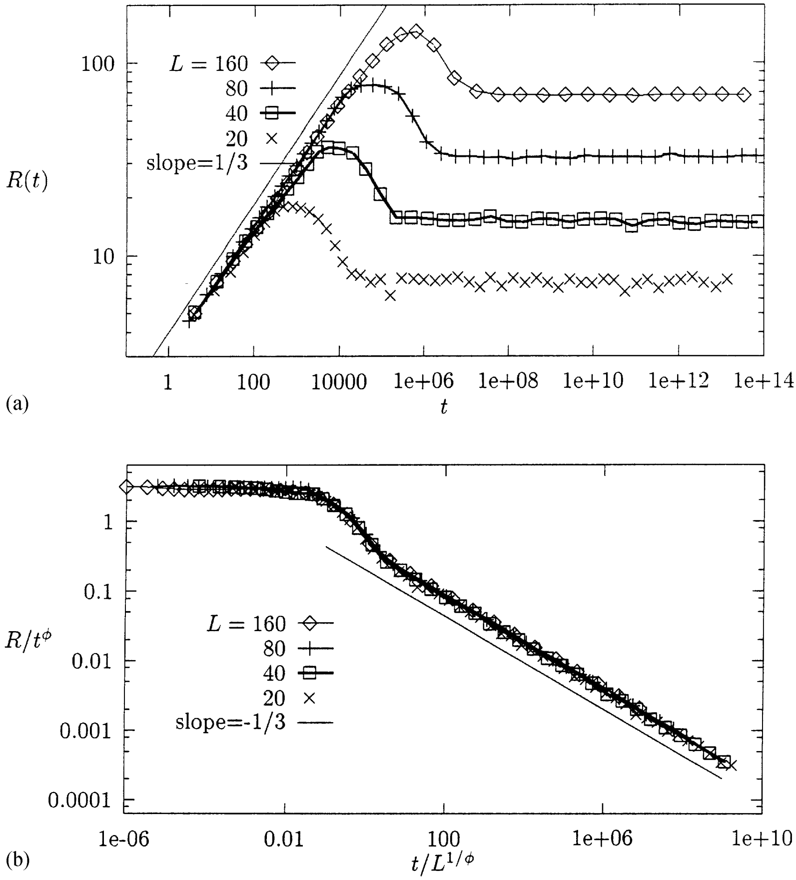

Fig. 4. (a) Finite-size effect shows that for the plateau $R_0(L_1)/R_0(L_2) \approx L_1/L_2$, which implies $R_0(L) \propto L$. $s = 0.5$, $A = 0.18$ with PBC. (b) Finite-size scaling plot of the same data.

Fortunately, there exist other more accessible, physically more meaningful measurements of avalanche size. Analogous to seismology [1], we define the "seismic moment" M as the slip distance summed over the "fault area" S:

$$M = K \sum_{l=1}^{S} u_l = \alpha \sum_{l=1}^{S} F_l^{\text{slip}} \gtrsim \alpha F_s S , \tag{9}$$

where the inequality follows from $F^{\text{slip}} \gtrsim F_s$. Furthermore, the "radiated seismic energy" E is given by the energy dissipated by friction over the fault area:

$$E = \frac{1}{2} \sum_{l=1}^{S} F_l^{\text{slip}} u_l = \frac{\alpha}{2K} \sum_{l=1}^{S} (F_l^{\text{slip}})^2 \gtrsim \frac{F_s M}{2K} , \tag{10}$$

where we have used Eqs. (8) and (9). These two quantities can be precisely retrieved from seismic monitoring. It is clear from Eqs. (9) and (10) that both the distribution

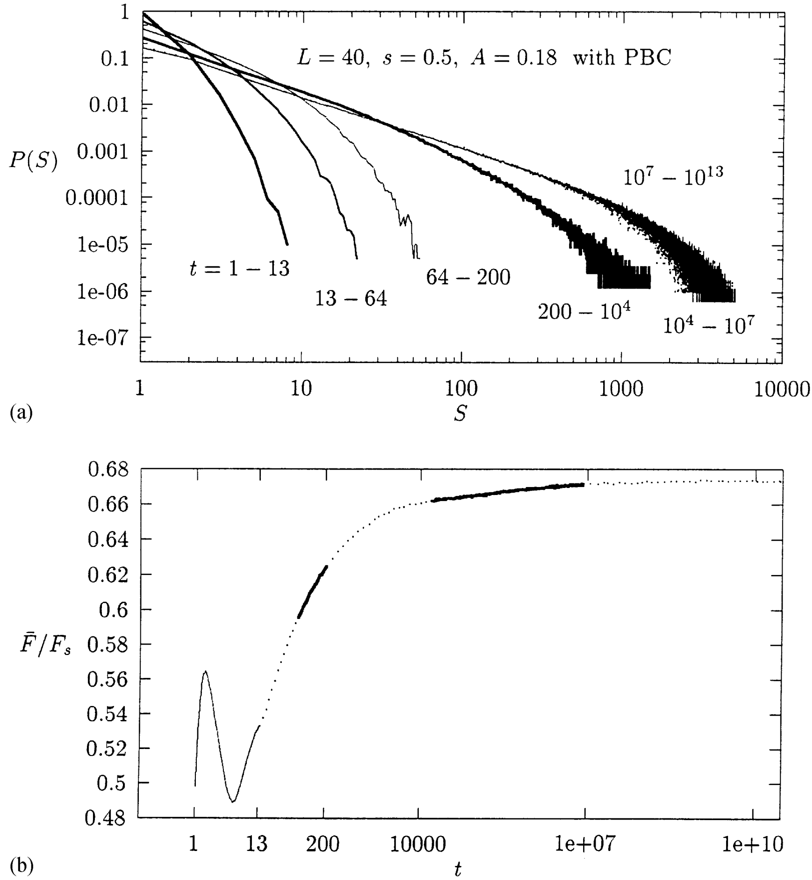

Fig. 5. (a) $P(S)$ obtained from successive intervals in t corresponding to the snapshots in Fig. 3, to show the approach to power law as $R(t) \rightarrow R_0$. (b) The distinct segments in this plot of $\bar{F}(t)/F_s$ vs. t correspond to the successive curves in (a).

of M and the *scaled* energy EK are stationary and follow the same power law as for S. Of course, the approach to an equilibrium state means that the dissipated energy gradually diminishes, and unlike a generic stationary state, *not every distributions are stationary*. But the punctuated manner of approach nevertheless leads to a marginally stable state characterized by stationary distributions of the above physical quantities.

One important feature that distinguishes our model from others is that the conserved quantities (F_x and F_y) are signed. Multiplicative loading does not increase them monotonically. In fact, due to Newton's third law, they sum over all sites to *zero at all times*. This is not the case for sandpiles [6] and other stick-slip models [16–19,7–11] where their dynamical variables are non-negative and increased additively by loading. This explains why those models are ill-defined when they are both conservative and periodic. Another way to rationalize our results is to notice that the mechanisms at work to produce power laws in the presence of multiplicative noises (amplification by multiplication followed by reinjection) [25] might also be relevant here.

4. Robustness tests

To gain more insight into the mechanism leading to SOC, we examine the robustness of the model against the following variations.

4.1. Scalar forcing

First we note that to our knowledge all previous stick-slip models have scalar forcing, i.e., the blocks moves only in one direction. In [20], a generalization to vectorial forcing was examined and found to be irrelevant in most cases. Similarly, the scalar version (e.g., $F_y \equiv 0$) of the present model is also found to exhibit the same properties as the vectorial version, thus belonging to the same universality class.

4.2. Boundary conditions

It is well known that boundary conditions are crucial to the properties of nonequilibrium states. For the OFC model with PBC, it flows into a periodic state characterized by avalanches of size one, where the sites topple in an order specified by the initial condition [9–11]. With free boundary conditions (FBC) instead, SOC state prevails with a power-law distribution of avalanche size. It has been suggested [11] that the existence of inhomogeneities at free edges, having α in Eq. (4) different than the bulk, serves as some kind of relevant perturbation which changes the nature of the fixed points of the dynamics from periodic to SOC. In stark contrast, we find for our model no quantitative difference of physical or statistical significance between PBC, FBC and a cylindrical boundary conditions (CBC, being half periodic and half free). In particular, their avalanche size distributions have the same exponent within numerical uncertainty (see Fig. 6). Although $R(t)$ grows linearly for FBC and CBC, it is not due to an invasion of SOC region, but rather due to the stress relief at free edges. SOC region remains to coarsen in the interior of the system with the same exponent, as studies of subsystems show [26].

4.3. Conservation

Next we explore the dependence on conservation laws. We introduce non-conservation by adding a term $-(x, y)\kappa K$ to (F_x, F_y) in Eq. (1), which may be thought of representing harmonic couplings of each block with another surface, such as a driving plate in earthquake models [16,17,7]. κ specifies the level of non-conservation, while α in Eq. (4) is now generalized to $1/(2 + 2s + \kappa)$ [7,20]. For $\kappa > 0$, our results show that the avalanche size distribution is exponential and the correlation length is

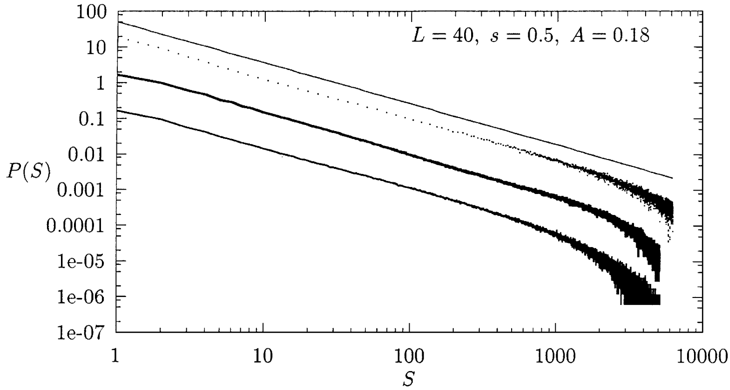

Fig. 6. Slip number distributions for different boundary conditions show the same exponent. From top to bottom: a slope of -1.15 for comparison, FBC, CBC and PBC. The curves for FBC and CBC have been shifted vertically for clarity.

finite. There are evidences [26] revealing a transition from a subcritical state (with truncated power-law distributions) to a strictly periodic state under PBC, whereas it is always subcritical under FBC. We conjecture that the system become critical only in the conservative limit $\kappa \to 0$, but the nature of this transition remains to be determined.

4.4. Quenched randomness

Further tests have been done [26], against the addition of randomly positioned, weakened springs of a fixed strength $0 \leqslant K_w < K$ and spatial density $0 < \rho < 1$. Remarkably, for any value of K_w and ρ, we find no deviation from the pure case in $P(S)$ apart from a smaller cutoff. Again this independence differs sharply from the effects of quenched randomness in other models [8,27].

5. Conclusion

To summarize, we have studied a new class of SOC models which is extremely robust (universal) with respect to the initial disorder, quenched disorder, initial strain, size of the system and the type of boundary conditions. It has been suggested that SOC in sandpiles and stick-slip models arises from an invasion mechanism initiated by inhomogeneities from free boundaries [11]. Our system reveals an alternative, without relying on such inhomogeneities, to attain criticality even with periodic boundaries. The mechanism involves building up correlation or SOC region *in the bulk* via a coarsening process.

The resulting characteristic dependence on system size in the convergence time $\tau \approx L^{1/\phi}$ (cf. Fig. 4b) contrasts dramatically with both the boundary invasion in the

OFC model and the usual exponential approach to equilibrium in thermodynamic systems away from critical points. This highlights the dual nature of our system as being critical and meanwhile approaching an equilibrium in the sweeping of the spring coefficient K. This teaches us that experimental systems appearing transiently driven might in fact be stationary in the variable relevant to the dynamics, especially when converging to a fundamental equilibrium state. Search schemes and optimization techniques using the sweeping of a control parameter such as in simulated annealing to get access to the fundamental state or to the optimal solution might exhibit this kind of phenomenon in which the relaxation is characterized by a wide distribution of jumps.

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