

## Dynamic exponent for 2D Ising model by power spectra method

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**Abstract.** We consider power spectra for fluctuating quantities at a critical point,  $P(\omega) \propto \omega^{-\varphi}$ , where  $\varphi$  is determined by the dynamic exponent  $z$  and static exponents specific to the quantity considered. From the magnetization spectra obtained with simulations,  $z$  is found to be  $2.13 \pm 0.03$  for the two-dimensional Glauber kinetic Ising model, hence in accord with recent Monte Carlo relaxational studies, but at odds with those obtained by series expansion and damage spreading methods. Excellent dynamic finite-size scaling for system sizes  $L = 10$  to 128 supports our estimate.

At the critical point of a second-order phase transition, the correlation length  $\xi$  generally diverges with the system size,  $L$ . In parallel to this static behaviour is the increase of the correlation time  $\tau \sim \xi^z$ , where  $z$  is the dynamic exponent. This is the well known phenomena called *critical slowing down* [1]. While the divergence of  $\xi$  arises from singularities of the free energy,  $\tau$  and  $z$  can only be determined from the dynamics [1], such as that described by a Langevin or a master equation. That the dynamics are much harder to solve explains why  $z$  is not exactly known except for certain one-dimensional (1D) models [2–5]. Computer simulation is therefore very valuable in providing a non-perturbative way of computing  $z$ .

In this paper, we will focus on the 2D Ising model. Despite considerable amount of works over the past two decades (see, e.g., [6, 7, 24, 26] and references therein), there is no universal agreement on its precise value. We are not going to give a comprehensive review on its present status, but merely to recall that its estimate varies widely: for instance, from  $2.076 \pm 0.005$  to  $2.34 \pm 0.03$ , deduced respectively from magnetization relaxation via Monte Carlo simulation [8] and series expansion [7]. These values bracket the commonly accepted estimate at about 2.16, obtained mainly by relaxational methods [6, 9, 10]. Recent studies using methods of damage spreading also add to the controversy [11, 12]; they tend to favour a large  $z \approx 2.3$ . Since different approaches continue to disagree with each other and some even produce mutually exclusive results, as is further evident from several latest works [21–26], it is desirable to have as many independent checks as possible.

Here we introduce a method which uses the power spectra to determine  $z$  from simulation data. The basic idea is simple: at a critical point, *temporal* scale invariance implies algebraic decay of correlations which is reflected as power laws in the power spectra of fluctuating quantities. This approach has been used [13] to re-derive  $z = 4 - \eta$  for the spin-exchange Ising model [14] (model B in the terminology of Hohenberg and Halperin [1]), where we

analysed the fluctuating average current. For the spin-flip kinetic Ising model [15] (i.e. model A [1]), we consider the following spectrum:

$$P(\omega) = \lim_{t_m \rightarrow \infty} \frac{1}{t_m} \left\langle \left| \sum_{t=1}^{t_m} m(t) e^{i\omega t} \right|^2 \right\rangle \quad (1)$$

where  $m$  is the average magnetization. In terms of the Fourier transforms, we have

$$P(\omega) = V^{-1} \tilde{G}(k=0, \omega) \quad (2)$$

where  $V = L^d$  is the volume of the system, and  $\tilde{G}$  is the correlation function in momentum-frequency space

$$\langle \tilde{m}(k, \omega) \tilde{m}(k', \omega') \rangle = V t_m \tilde{G}(k, \omega) \delta_{k, -k'} \delta_{\omega, -\omega'}. \quad (3)$$

Since  $\tilde{G}(k, \omega) = k^{-2+\eta-z} g(k^{-z} \omega)$  is established non-perturbatively by dynamic renormalization group methods [16, 1], we obtain by requiring  $\tilde{G}$  to be finite as  $k \rightarrow 0$  for finite  $\omega$

$$P(\omega) \sim V^{-1} \omega^{-\varphi} \quad (4)$$

where  $\varphi = 1 + (2 - \eta)/z$ . In passing, note that this procedure fails for model B in the presence of a discontinuity at  $k = 0$ , due to the conservation of  $m$ . Thus the simplest spectrum to consider is the current instead [13].

For finite  $L$  and finite  $t_m$  (=length of time series), we generalize (4) by making a finite-size scaling ansatz [17]

$$P(\omega, L, t_m) = L^{-d} \omega^{-\varphi} G(\omega L^z, t_m L^{-z}). \quad (5)$$

Ignoring the argument  $t_m/L^z$  for the moment, the scaling function satisfies asymptotically

$$G \sim \begin{cases} \text{constant} & \omega L^z \gg 1 \\ (\omega L^z)^\varphi & \omega L^z \ll 1 \end{cases} \quad (6)$$

because for 'observation time'  $\omega^{-1} \ll \tau_L \propto L^z$ ,  $P$  knows no  $L$  (except the trivial overall  $L^{-d}$ ), and in the opposite limit  $P$  must be finite:  $P(\omega \rightarrow 0, L) \sim L^{2-\eta+z-d}$ . The latter is realized if the time series are long enough such that the minimum frequency  $2\pi/t_m \ll L^{-z}$ . Equation (5) can also be rewritten in the equivalent form

$$P(\omega, L, t_m) = L^{2-\eta+z-d} \tilde{G}(\omega L^z, t_m L^{-z}). \quad (7)$$

Now we discuss the effect of  $t_m/L^z$  in  $G$  or  $\tilde{G}$  above, which is introduced on dimensional ground. In principle, unless this argument is kept constant, data for different  $L$  will not collapse in a plot of  $L^{-2+\eta-z+d} P$  versus  $\omega L^z$ , so there will be no such simple visualization of dynamic finite-size scaling. In practice, since we expect  $G$  to be analytic in the limit  $G(\omega L^z, t_m/L^z \rightarrow \infty)$ , the effect of  $t_m/L^z$  is negligible for sufficiently long time series. This is possible for small  $L$ , but not so for larger  $L$ , as the length  $t_m$  is increasingly limited by computing power.

We have done Monte Carlo simulations for the 2D Ising model at the critical point, on a periodic square lattice, using single spin-flip algorithm [15]. Figure 1 shows the power spectra for the magnetization, where  $t_m = 32768$  for  $L \leq 64$ , 4096 for  $L = 80$  and 2048 for  $L = 128$ . The data represent averages over 1000 runs to ensure good statistics. From the slopes of these curves, we immediately get  $z$  (see figure 2). The best estimate is  $z = 2.13 \pm 0.03$  from  $L = 128$ . Including errors, this is somewhat smaller but agrees with estimates by relaxational methods [9, 6, 10, 21, 26].

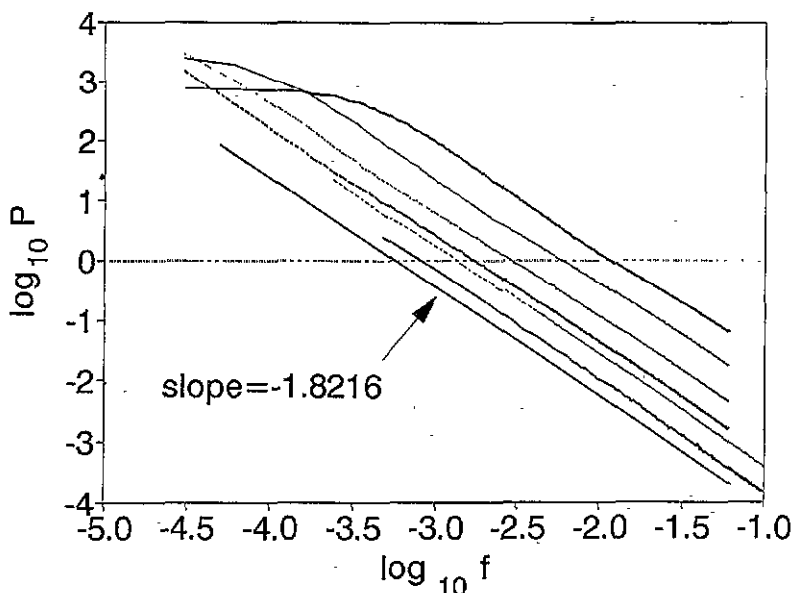


Figure 1. Log-log plot of power spectra for magnetization of 2D Ising model with single-spin-flip algorithm.  $f = \omega/2\pi$ , system size  $L = 10, 20, 40, 64, 80$  and  $128$  from top down. The bottom straight line is for reference. Its slope corresponds to  $z = 2.13$ .

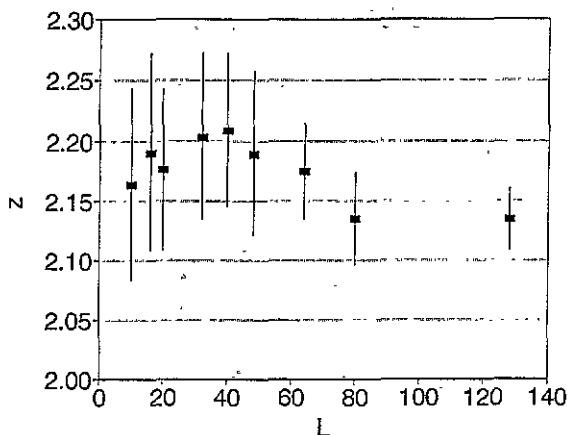


Figure 2. Estimates of dynamic exponent  $z$  from the slope of power spectra in figure 1, using equation (4).

Regarding finite-size scaling, the problem with finite  $t_m/L^z$  mentioned above is apparent in figure 3, which shows the systematic effect of varying  $t_m$  with fixed  $L$ . By adjusting the relative vertical position of these curves, we find complete overlap. Therefore, aside from vertical shifts, the curves with longer  $t_m$  appear to be extensions of those with shorter  $t_m$  to smaller frequencies. Since we are not aware of any theoretical prediction for the dependence of  $P$  on  $t_m/L^z$ , the above statement will be taken as an empirical observation. For practical reason, our main data in figure 1 actually have shorter  $t_m$  for larger  $L$ . To circumvent the resulted deviations from simple scaling, a separate reference set of data consisting of shorter  $t_m$  were taken with fixed  $t_m/L^z$ . Then the log  $P$  curve for each  $L$  in figure 1 was shifted to

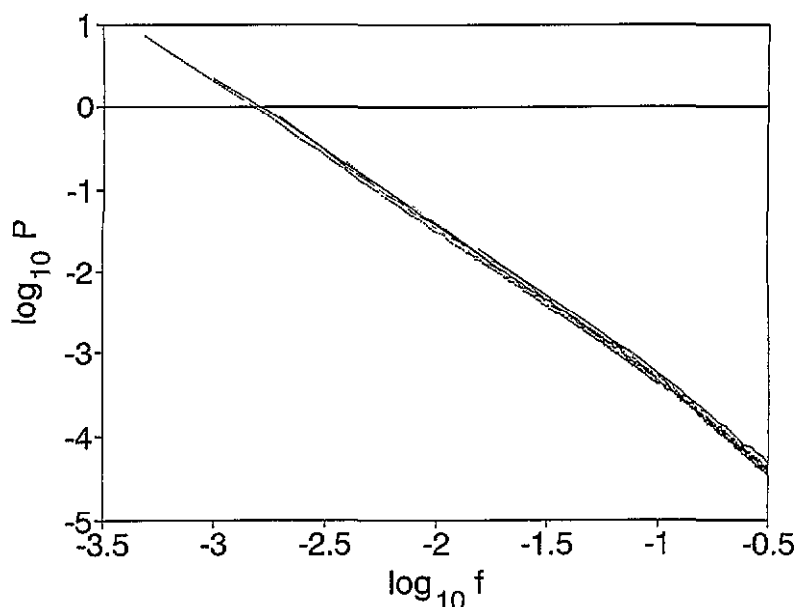


Figure 3. Finite  $t_m/L^2$  causes small, systematic shift in  $P$ . Data are from multispin coding algorithm,  $L = 128$ , and from top to bottom:  $t_m = 64, 128, 256, 512, 1024$  and  $2048$ .

match with that reference set of data. This corrective procedure effectively suppresses the dependence of  $\bar{G}$  on  $t_m/L^2$  in (7), and leads to excellent dynamic scaling, as exhibited in figure 4.

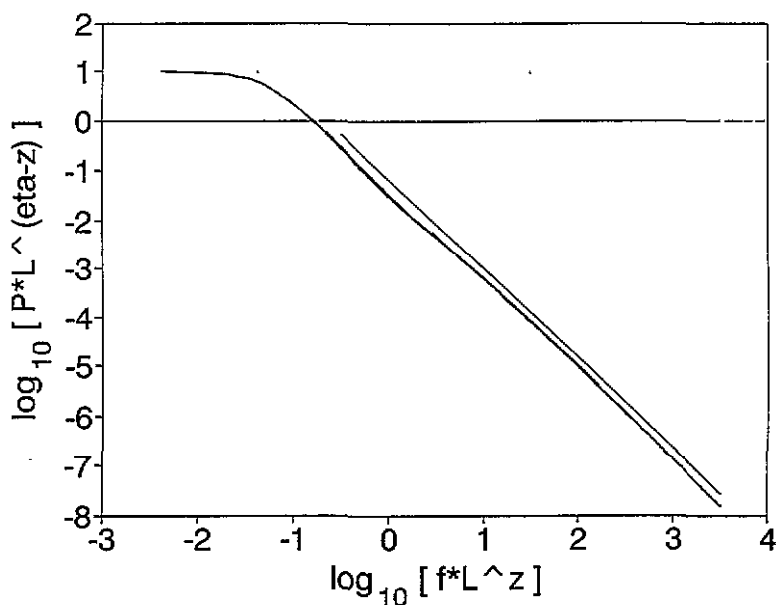


Figure 4. Dynamic finite-size scaling for the data that use single-spin-flip algorithm (from figure 1). The line of reference has a slope corresponding to  $z \approx 2.13$  which gives the best overlap.

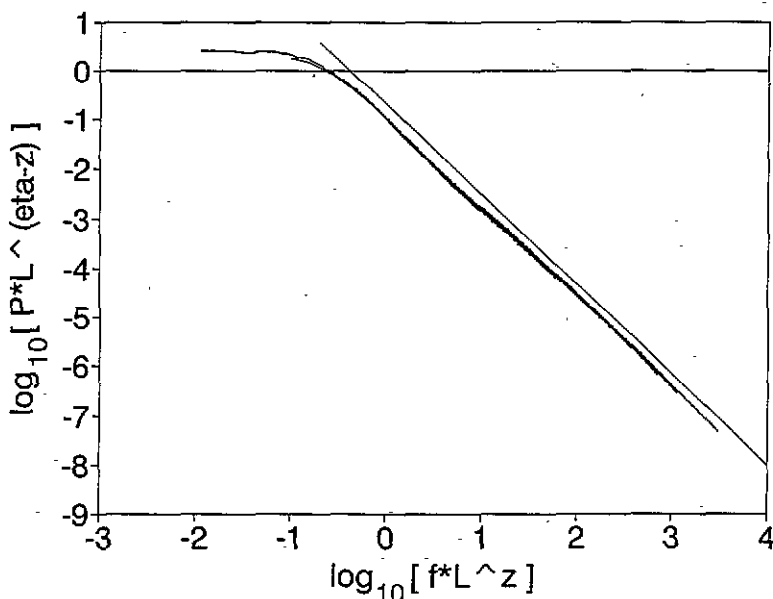


Figure 5. Dynamic finite-size scaling for  $L = 16$  to  $128$ , using multispin coding algorithm.  $z = 2.13$  is used. Up to a shift (equation (8)), the scaling function appears indistinguishable from that in figure 4, indicating the same universality class.

Several comments are in order: first, we emphasize that the deviations prior to corrections are very small, and will not be resolved given poorer statistics. Even for small  $t_m/L^z$  where we expect problem, the slope appears to be unaffected (see figure 3) so that  $z$  can still be extracted. In any case, such a procedure of shifts is unnecessary if one only chooses  $t_m \propto L^z$  (with reasonable trial  $z$ ). Second, as shown in figure 4, the scaling function  $\bar{G}$  has an inflection point between the two asymptotic limits. This results in finite-size effects well before the plateau is reached. It is the origin of the large errors for small  $L$  in figure 2. Third, there are expected systematic deviations from scaling for data at high frequencies which are not shown in the figures for clarity. Such high- $f$  tails tell us precisely where the temporal scaling regime as described by (5) begins.

We have also obtained a separate set of data using the multispin coding technique [19, 20], where an entire sublattice is updated at once. As anticipated, the system evolves faster

$$\tau_L^{(\text{multispin})} = \tau_L^{(1\text{-spin})}/c$$

where numerically we find  $c \approx 3.84$ . The resulted spectra is related to those of single-spin-flip algorithm via a simple rescaling (cf [18])

$$P^{(\text{multispin})}(\omega, L, t_m) = c^{-1} P^{(1\text{-spin})}(\omega/c, L, ct_m) \quad (8)$$

where the prefactor ( $c^{-1}$ ) ensures the same statics for both algorithms. Other than an overall change in time scale, the two belong to the same dynamic universality class and give the same estimates of  $z$  (see figure 5).

Our approach has certain advantages: it is clean and precise, giving an estimate of  $z$  from runs for one large system size, with no need of extracting  $\tau_L$  explicitly and studying its finite-size effects. Furthermore, the derivation works for quantities in addition to magnetization. For example, one can show that the spectrum for the mean energy also obeys (5) but with

$\phi = 1 + \alpha/\nu z$ . Of course, the 2D Ising model is a special case here because the fluctuation of energy is logarithmic, and it is not clear how logarithmics should be incorporated into the finite-size scaling ansatz. Nevertheless, we get fine data collapse analogous to figure 4, with  $\alpha = 0$ . This alternative will be useful for models where  $\alpha/\nu$ , instead of  $\eta = 2 - \gamma/\nu$ , is available. Finally, since our method relies only on the validity of the scaling form for correlation functions (below equation (3)), applications to higher dimensions and to other models are immediate, investigations already being under way.

In conclusion, in parallel to the treatment for conserved order parameter [13], we have shown how the exponent  $z$  can also be determined precisely by power spectra for the case of non-conserved order parameter. Using the 2D Glauber Ising model as an example, we find  $z = 2.13 \pm 0.03$ , somewhat smaller but in agreement with those obtained by the relaxational methods [6, 9, 10, 21, 26]. Thus an intermediate value among current estimates is favoured.

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*Note added.* After the completion of this work, we received a preprint by Lauritsen and Fogedby [27], who have used the same method to compute  $z$  for the 2D and 3D Ising models. Although their statistics are insufficient to show the correction of scaling from the presence of  $t_m/L^z$ , their estimate for the 2D Ising model is consistent with ours. They have also analysed power spectra for certain models of interface growth and sandpiles, and obtained some useful scaling relations.

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