

## Tricritical Behavior in Rupture Induced by Disorder

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We discover a qualitatively new behavior for systems where the load transfer has limiting stress amplification as in real fiber composites. We find that the disorder is a relevant field leading to tricriticality, separating a first-order regime where rupture occurs without significant precursors from a second-order regime where the macroscopic elastic coefficient exhibits power law behavior. Our results are based on analytical analysis of fiber bundle models and numerical simulations of a two-dimensional tensorial spring-block system in which stick-slip motion and fracture compete. [S0031-9007(97)02698-7]

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There is growing evidence that rupture in random media can be viewed as a kind of critical phenomenon [1,2] as a result of the interplay between disorder and fracture mechanics, with proposed applications, in particular, to fiber composites [3,4] and earthquakes [5]. Notwithstanding its importance, we do not have a comprehensive understanding of rupture phenomena but only a partial classification [1,2,6]. From a theoretical point of view, rupture is controlled in principle by the infinite moment  $\langle \sigma^q \rangle|_{q \rightarrow \infty}$  of the stress field and the difficulties emerge from the noncommutation of the two limits ( $q \rightarrow \infty, \Delta \rightarrow 0$ ), where  $\Delta$  is the amount of disorder (see below for a precise definition). In intuitive wording, the largest stress in the system is very sensitive to the amount and type of disorder. Disorder is known to induce stress field distributions with fat tails [7]. Consider, for instance, a log-normal distribution with standard deviation  $\Delta$  and mean  $\sigma_0$ , then  $\langle \sigma^q \rangle^{1/q} = \sigma_0 e^{\Delta^2 q/2}$ , which shows that the limit  $\Delta \rightarrow 0$  is singular for rupture ( $q \rightarrow \infty$ ). This noncommutativity of limits is at the crux of some of the major outstanding problems in physics [8] such as turbulence (viscosity  $\rightarrow 0$ ; time  $\rightarrow \infty$ ) and quantum chaos ( $\hbar \rightarrow 0$ ; time  $\rightarrow \infty$ ). Here, we show that the amount of disorder  $\Delta$  plays the role of a relevant field which makes systems with limited stress amplification exhibit a tricritical transition as the disorder increases, from a Griffith-type abrupt rupture (first-order) regime to a progressive damage (critical) regime. This is reminiscent of the critical behavior induced by quenched disorder in magnetic systems [9].

We first document this behavior in a simple mean-field model of rupture, known as the democratic fiber bundle model [10]. It consists of  $N$  parallel fibers with identical spring constants and identically independent random failure thresholds  $X_j$  distributed according to the cumulative probability distribution  $P(X_j < x) \equiv P(x)$ . A total force  $F$  is applied to the system and is shared democratically among the  $N$  fibers. When the force on one fiber reaches its threshold, the fiber ruptures and the stress is redi-

tributed to all remaining fibers. This transfer might induce secondary failures which in turn induce tertiary ruptures and so on. One is interested in the stress-strain characteristic as the applied force is increased, the properties of the rupture point and the precursory events prior to the complete breakdown. The solution of this problem is found by noticing that the total bundle will not break under a load  $F$  if there are  $n$  fibers in the bundle, each of which can withstand  $x_n \equiv F/n$ .  $x_n$  and  $n$  are related, for large  $N$ , by  $n = N[1 - P(x_n)]$  leading to  $F(x_n) = Nx_n[1 - P(x_n)]$ . The number  $k$  of fibers which have failed under the force  $F$  is then  $k = N - n = NP(x_n)$ . Now, for a broad class of distribution  $P(x)$  extending down to 0, the function  $x[1 - P(x)]$  presents a maximum at  $0 < x^* < \infty$ , the solution of  $dx[1 - P(x)]/dx|_{x=x^*} = 0$ . As the behavior of  $F(x_n)$  close to  $x^*$  is quadratic  $F(x_n) \approx F^* - c(x^* - x_n)^2$  where  $c$  is a constant, this implies that the rate  $dk/dF$  of fiber failure diverges as  $(F^* - F)^{-1/2}$ , where  $F^* = x^*[1 - P(x^*)]$ , thus qualifying a critical mean-field behavior.

However, if  $P(x)$  is such that  $dx[1 - P(x)]/dx = 0$  has no solution, the behavior will be completely different, with a sequence of a few fibers maybe breaking as the load is applied followed by an abrupt global failure. Correspondingly, the stress-strain characteristic exhibits a discontinuity in its slope at the point of rupture. This qualifies a first-order behavior. Notice that this is similar to the Ehrenfest's classification of the order of phase transitions, where the free energy is here replaced by the elastic energy.

Let us take, for instance,  $P(x) = 0$  for  $0 \leq x < x_1$ ,  $P(x) = (x - x_1)/\Delta$  for  $x_1 \leq x \leq x_1 + \Delta$ , and  $P(x) = 1$  for  $x \geq x_1 + \Delta$ , corresponding to the strengths  $X_j$  uniformly distributed between  $x_1 > 0$  and  $x_1 + \Delta$ . Then,  $dx[1 - P(x)]/dx = (x_1 + \Delta - 2x)/\Delta$ , which has a root in the interval  $x_1 \leq x \leq x_1 + \Delta$  if and only if  $\Delta > x_1$ . In this case, we recover the previous mean-field critical behavior. However, for weak disorder  $\Delta < x_1$ , not a

single fiber breaks down until the force reaches  $Nx_1$  at which value the system of  $N$  fibers breaks suddenly. This is an extreme illustration of a “first-order” behavior. The particular value  $\Delta = x_1, F = Nx_1$  thus plays the role of a tricritical point in analogy with thermal phase transitions [11]. This behavior holds for a large class of distributions  $P(x)$ : the condition that  $dx[1 - P(x)]/dx$  has no root is equivalent to the condition that the equation  $d \ln[1 - P(x)]/dx = -1/x$  has no solutions for any  $0 \leq x < \infty$ . This equation defines two domains: (1)  $d \ln[1 - P(x)]/dx < -1/x$  for all  $x \geq 0$ : this can occur, in particular, if  $1 - P(x)$  decays to zero faster than  $1/x$  for large  $x$ , with the additional constraint that there exists a minimum strength  $x_1$  strictly positive. Notice that the distributions which extend down to zero are in this sense always in the “large” disorder regime. (2)  $d \ln[1 - P(x)]/dx > -1/x$  for all  $x \geq 0$ : this corresponds to distributions which decay slower than  $1/x$ . Take, for instance,  $1 - P(x) = (1 + x)^{-\alpha}$ . Then,  $d \ln[1 - P(x)]/dx = -\alpha/(1 + x)$  which remains strictly larger than  $-1/x$  if  $\alpha < 1$ .

Having established the existence of the tricritical behavior in this mean-field model, let us now turn our attention to a more realistic two-dimensional (2D) spring-

block model of surface fracture in which the stress can be released by spring breaks *and* block slips. We consider the experimental situation where a balloon covered with paint or dry resin is progressively inflated [12]. An industrial application is a metallic tank with carbon or kevlar fibers impregnated in a resin matrix wrapped up around it which is slowly pressurized [3]. As a consequence, it elastically deforms, transferring tensile stress to the overlayer. Slipping (called fiber-metal delamination) and cracking can thus occur in the overlayer. We model this process by a 2D array of blocks which represents the overlayer on a coarse grained scale in contact with a surface with solid friction contact. The solid friction will limit stress amplification. Each block is interconnected to its nearest neighbors via springs of unstretched lengths  $l_0$  and spring constants  $K$ . The position of the blocks in the  $x$  and  $y$  directions are given by  $(ai + x_{i,j}, aj + y_{i,j})$  where  $1 \leq i, j \leq L$  form a square lattice with lattice constants  $a$ , and where  $x_{i,j}, y_{i,j}$  fulfill  $x_{i,j}, y_{i,j} \ll a$ , so that Hooke’s law applies. In [13] it was shown that the  $x$  component of the force on a block *to first order* in the displacements takes the form

$$F_{i,j}^x = -K\{(b_{i+1,j} + b_{i-1,j})x_{i,j} - b_{i+1,j}x_{i+1,j} - b_{i-1,j}x_{i-1,j} + s[(b_{i,j+1} + b_{i,j-1})x_{i,j} - b_{i,j-1}x_{i,j-1} - b_{i,j+1}x_{i,j+1}] - as(b_{i+1,j} - b_{i-1,j})\}, \quad (1)$$

and, by symmetry,  $F_{i,j}^y$  follows by switching  $x \leftrightarrow y$  and  $i \leftrightarrow j$ .  $s \equiv (a - l_0)/a \geq 0$  is the strain of the network without fluctuations ( $x_{i,j}, y_{i,j} \equiv 0$ ), and  $b_{i \pm 1, j \pm 1} = 1, 0$ , respectively, depending on whether a spring connects the blocks  $(i, j) - (i \pm 1, j \pm 1)$  or not. Likewise the stress  $B$  in a spring is given by

$$B_{(i,j)-(i \pm 1, j)} = K[(x_{i,j} - x_{i \pm 1, j} - sa)^2 + s^2(y_{i,j} - y_{i \pm 1, j})^2]^{1/2}. \quad (2)$$

Initially  $x_{i,j}$  and  $y_{i,j}$  are chosen uniformly from the interval  $[-\Delta, +\Delta]$ , thus  $\Delta$  quantifies the amount of disorder which is on the initial displacements corresponding to an effective initial disorder in the thresholds. Periodic boundary conditions are used in both the  $x$  and  $y$  directions. This corresponds to a torus topology, which is an adequate approximation of a cylindrical tank, as both are characterized by the absence of flexion. A spherical topology has flexion, and we would thus need to enrich the model by adding flexure deformation in the model, for instance, by replacing the springs by beams. However, we do not expect this to modify our results as the dimensionality of the model does not influence qualitatively the transition, as seen in the mean-field fiber bundle model.

The coupling of the overlayer to the substrate has two effects when the substrate expands: (1) Tensile stress is transferred to the overlayer. This is taken into account by imposing an increase in the average distance  $a$  between the blocks so as to reflect the inflation of the balloon. As a definition of the time,  $t$ , we let  $a(t)$  increase linearly

with  $t$ . (2) The increasing tensile stress in turn gives rise to stick-and-slip motion or/and cracking. A block is assumed to stick until the total force applied on it exceeds a threshold  $F_s$ , where after it slips to the zero-force position, corresponding to local mechanical equilibrium in the absence of friction. This thereby releases stresses on its neighbor blocks. A spring breaks irreversibly once the stress  $B$  exceeds a threshold  $F_c \equiv \kappa F_s$  [14].

We define a time dependent apparent macroscopic stress on the system,  $\sigma_{\text{app}}(t)$ , from the relation  $\sigma_{\text{app}}(t) = E(t)/\epsilon(t)$ , where  $E(t)$  is the total elastic energy stored in the springs of the system at time  $t$ , and  $\epsilon(t) \equiv a(t)/a(t=0)$  is the macroscopic strain. We calculate an effective Young modulus, given by

$$Y_{\text{app}}(t) \equiv d\sigma_{\text{app}}(t)/d\epsilon. \quad (3)$$

$Y_{\text{app}}(t)$  can be expressed as  $[d\sigma_{\text{app}}(t)/d\sigma(t)](d\sigma/d\epsilon)$ , where  $\sigma$  is proportional to the first invariant (the trace) of the real stress field in the system.  $d\sigma/d\epsilon$  is the corresponding elastic modulus expected to exhibit a power law behavior if criticality is present, while  $d\sigma_{\text{app}}(t)/d\sigma(t)$  goes to a constant. Therefore, the measurement of  $Y_{\text{app}}(t)$  gives us direct access, if present, to the critical behavior of the Young modulus of the system. As the strain  $\epsilon(t)$  is increased, block slips and spring failures occur up to a point where the system is completely ruptured and the stress necessary to impose a constant small strain rate starts to decrease from a maximum. At this point, there is at least

one large crack spanning the whole system. If global rupture occurs abruptly (first-order case),  $\sigma_{\text{app}}(t)$  must exhibit a sharp maximum and  $Y_{\text{app}}(t)$  remains finite. If, on the other hand, the rupture is critical,  $\sigma_{\text{app}}(t)$  will exhibit a progressive rounding with a smooth maximum, while  $Y_{\text{app}}(t)$  vanishes as a power law  $Y_{\text{app}} \propto [(\epsilon_c - \epsilon)/\epsilon_c]^\gamma$  on the approach to rupture at  $\epsilon_c$ .

In Fig. 1 are shown the stress-strain curves from one single realization for each different system size, with  $\Delta = 0.75\Delta_c$ , where  $\Delta_c$  is the maximal amplitude such that at  $t=0$   $B_{(i,j)-(i\pm 1,j\pm 1)} < F_c$  and  $|\vec{F}_{i,j}| < F_s$  for all  $(i, j)$ .  $\Delta_c$  has been determined numerically for each run. We set  $a(t=0) = l_0 = 1$  and  $F_s/K = 1$  throughout this paper. Observe that the maximum stress a system can sustain is an increasing function of  $\kappa$  and the range of  $\epsilon$  values over which fracturing takes place decreases as  $\kappa$  increases. For  $\Delta = 0.75\Delta_c$  and  $\kappa < 2.9$ , the stress-strain curve presents a smooth maximum indicating a critical rupture. This is confirmed in Fig. 2, showing the vanishing of the apparent Young modulus  $Y_{\text{app}}$  as  $\epsilon \rightarrow \epsilon_c$  from below. Each curve is obtained by averaging over  $N = 1000-5000$  independent configurations with system size  $L = 30$ .  $\epsilon_c$  has been estimated from the condition  $d\langle\sigma_{\text{app}}\rangle/d\epsilon|_{\epsilon=\epsilon_c} = 0$ , where  $\langle\cdots\rangle$  stands for an ensemble average. Larger lattice sizes ( $L = 50-400$ ) with the same value of  $LN$  were used, but for a given fixed  $LN$  we found the smallest lattice sizes  $L (= 30)$  give the best statistics, which we attribute to the lack of self averaging. For small  $\kappa$ , the exponent  $\gamma$  approaches a value slightly larger than 1, while it decreases continuously to zero as  $\kappa$  increases, as shown in the inset. It seems to vanish around  $\kappa = 2.9$ , signaling the transformation of the critical regime to an abrupt “first-order” behavior. Keeping  $\kappa$  fixed and varying  $\Delta/\Delta_c$ , we find that  $\gamma$  stays constant, but the size of the critical region increases with the magnitude of  $\Delta/\Delta_c$ . It shrinks to zero at a threshold value function of  $\kappa$  which is shown in Fig. 3. This function gives the boundary in the  $(\Delta/\Delta_c; \kappa)$  plane between the critical and first-order regime. As announced, for fixed  $\kappa < 2.9$ , increasing the disorder  $\Delta$  allows the system to go from a first-order to a critical regime. The

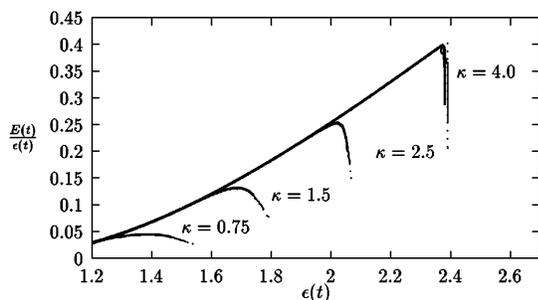


FIG. 1. Stress-strain curves for different values of  $\kappa$  and for different system sizes  $L = 50$  (dotted line), 100 (thin bold line), and 200 (bold line). The fracturing is stopped at different  $\epsilon(t)$  for different  $L$  in order to distinguish between the curves. (Notice that due to data collapse, the data for the various  $L$  become indistinguishable for  $\kappa = 0.75, 1.5, 2.5$ ). One  $L = 400$  simulation has been done for  $\kappa = 4$  (thick bold line).

fact that the disorder is so relevant as to create the analog of a tricritical behavior can be tracked back to the existence of solid friction on the blocks which ensures that the elastic forces in the springs are carried over a bounded distance (equal to the size of a slipping “avalanche”) during the stress transfer induced by block motions.

When  $\kappa$  is large, the system responds initially to an expansion by the release of stress uniquely by block slips. The block slips give rise to a stress rearrangement, and a spatial coarsening phenomenon of the stress field  $B$  takes place [15]. The enhanced correlations in the  $B$  field result in turn in a coherence when fracturing sets in, amounting to smoothing out the disorder, thus allowing for a large crack to develop in an abrupt way. For sufficiently large  $\kappa$ , the system breaks into two parts. This is the regime of first-order behavior. Notice that increasing  $\kappa$  in the  $(\Delta/\Delta_c; \kappa)$  phase diagram corresponds to decreasing the disorder and changing at the same time the distribution of disorder so that it becomes more correlated. This is therefore a more complicated route than just changing the width of the threshold distribution as in the previous fiber bundle model.

The value of the Young critical exponent  $\gamma$  for small  $\kappa$  can be predicted from percolation theory. Indeed, consider the limit  $\kappa \rightarrow 0$ , for which the blocks are stuck to the substrate and cannot move. Only the springs can fail, and they do so in a completely uncorrelated way, controlled by the initial random configuration of the blocks. We thus get an uncorrelated random dilution, ending at the percolation threshold where a macroscopic crack spans the system. In the presence of internal strain, it was shown that the elastic constant decreases to zero when the dilution increases with an exponent given by the *scalar* elasticity problem [16], equal to the conductance exponent of percolation. Extensive numerical simulations give the value 1.300 for this exponent [17], which is in agreement with the extrapolation of our results for  $\kappa \rightarrow 0$ .

It is important to understand that these properties belong to systems with load transfer mechanisms limiting

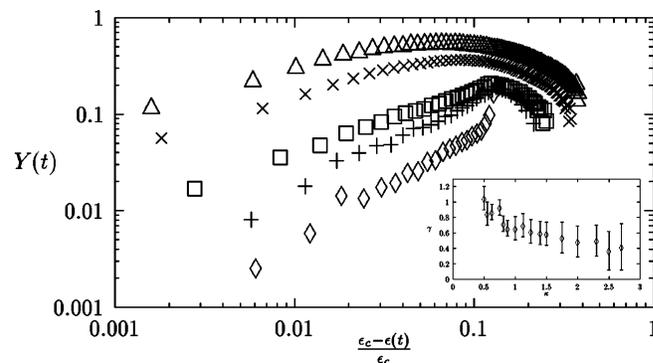


FIG. 2. Macroscopic Young modulus vs reduced macroscopic strain for different values of  $\kappa = 0.5(\diamond)$ ,  $0.75(+)$ ,  $0.875(\square)$ ,  $1.4(\times)$ , and  $2.0(\triangle)$ .  $\Delta/\Delta_c = 0.75$ . The inset shows the exponent  $\gamma$  as a function of  $\kappa$ .  $\kappa \approx 0.5$  is the smallest value for which the system initially has no bonds that exceed the threshold.

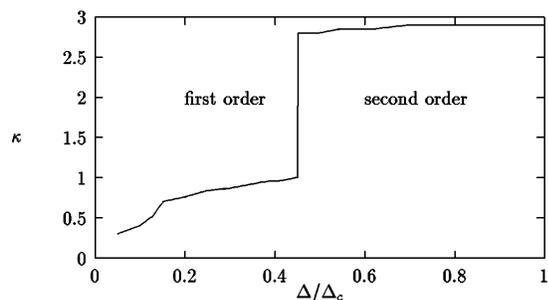


FIG. 3. Phase diagram for criticality of the fracturing.  $\Delta/\Delta_c \rightarrow 0$  cannot be studied within the spring-block model since ambiguity in the updating rules for stress release affects the fracturing.

stress amplification at crack tips. If no coupling or delay mechanism exists to regularize the divergence induced by elasticity at the crack tips (with a stress diverging as  $1/\sqrt{r}$  next to a crack tip), the first-order behavior is only observed for zero disorder described by the single-crack Griffith criterion and any amount of disorder is relevant to produce a critical behavior [1,2]. However, even in this case, the amount of disorder remains of utmost importance as it controls the size of the critical region, and therefore its observability [18].

Extensive simulations exploring the large parameter space ( $K, F_s, F_c$ ) and their respective disorder distribution, with the addition of cohesion, will allow us to classify the qualitative features of concrete materials parameters, such as fiber matrix cohesion, fiber tensile strength, and fiber matrix order, etc., which would be required for a system to fall within the first (respective, second) order regime. This will be reported elsewhere [19]. The existence of different regimes for rupture, depending on the limiting stress amplification and on disorder, opens the road to important potential applications for failure prediction purposes such as in the time-to-failure approach [20]. We suggest that the often observed power law distribution of acoustic emission bursts of many materials upon stressing offers an additional evidence of the critical nature of the damage and cracking of heterogeneous materials [3,21]. Our results provide the foundation for understanding why some systems exhibit clearer precursors before rupture than others in which they may even be absent in certain cases and for quantifying the expected amount and style of precursory activity as a function of heterogeneity and range of interaction.

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