Self-Organized Criticality in Stick-Slip Models with Periodic Boundaries

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A spring-block model governed by threshold dynamics and driven by temporally increasing spring constants is investigated. Because of its novel multiplicative driving, criticality occurs even with periodic boundary conditions via a mechanism distinct from that of previous models. This mechanism is dictated by a coarsening process. The results show a high degree of universality. The observed behavior should be relevant to a class of systems approaching equilibrium via a punctuated threshold dynamics. [S0031-9007(98)05441-6]

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Out-of-equilibrium driven systems with threshold dynamics exhibit a rich phenomenology, from synchronized behavior [1,2] to self-organized criticality (SOC) [3–5]. SOC refers to the spontaneous organization towards a kind of dynamical critical steady state. Threshold out-of-equilibrium dynamics encompasses many systems, such as neural networks, solid friction, rupture with healing, earthquakes, and avalanches. It is now understood qualitatively that there is a class of models exhibiting SOC as a result of their tendency to synchronize [6]. This tendency is, however, frustrated by constraints such as open boundary conditions [5,6] and quenched disorder [7] which lead to a dynamical regime at the edge of synchronization, the SOC state. Another class, so-called extremal models, is understood to exhibit SOC due to the competition between local strengthening and weakening due to interactions [8]. In a third class of models, SOC results from the tuning of the order parameter of a system exhibiting a genuine critical point to a vanishingly small, but positive value, thus ensuring that the corresponding control parameter lies exactly at its critical value for the underlying depinning transition [9]. The issue is furthermore complicated by the fact that a notable fraction of numerical and experimental works [10–12] claiming the observation of SOC from the measurements of power-law distributions rely on the slow sweeping of a control parameter towards a critical point [13,14].

The purpose of this Letter is to present a variation of spring-block models using a novel form of driving. The surprising result is that, when the dynamics is coupled to conservation laws and for *periodic boundary conditions* (PBC), the system self-organizes into a critical state with stationary distributions, despite approaching equilibrium. None of the four above mechanisms seems at work here, especially in dramatic contrast to other stick-slip models [4,5,15] which require either open boundaries and/or dissipation to desynchronize and hence achieve SOC. To our knowledge, our model is the first of its kind to show SOC in approach to an equilibrium state, and as such suggests a new class of experimental systems which could exhibit SOC states.

Model.—We consider a two-dimensional spring-block model consisting of an array of blocks interconnected among nearest neighbors by coil springs. The springs have the same spring constant K and relaxed spring length l. Initially, the array is stretched to a lattice spacing $a > l$ and placed on a frictional substrate which is characterized by a static threshold F_s for slipping. Disorder is introduced in the form of random displacements (x, y) of the blocks about the coordinates (ia, ja) on a square lattice, where $-A \le x, y \le A$, and $i, j = 1, \dots, L$. The force between two neighboring blocks at $\vec{r}$ and $\vec{r}'$ is given by Hooke's law $(|\vec{r} - \vec{r}'| - l)K$. Since we are interested in dynamics primarily governed by tensile stresses, this nonlinear dependence on the coordinates leads to unnecessary complications in the algorithm. To simplify and compare with similar models, we expand the expression to first order in (x, y) to obtain the force components on a block in the bulk at (i, j) [16],

$$\begin{aligned} F^x_{i,j} &= (x_{i+1,j} + x_{i-1,j} - 2x_{i,j})K \\ &\quad + (x_{i,j+1} + x_{i,j-1} - 2x_{i,j})sK\,, \\ F^y_{i,j} &= (y_{i+1,j} + y_{i-1,j} - 2y_{i,j})sK \\ &\quad + (y_{i,j+1} + y_{i,j-1} - 2y_{i,j})K\,, \end{aligned} \tag{1}$$

where $s = 1 - l/a > 0$ is the initial strain. It is important to stress that the terms proportional to s lead to anisotropic couplings to nearest neighbors in the transverse direction. The coupling and the SOC state disappear for $s = 0$ or in 1D chains.

Since the forces are linear in displacements, it is possible to invert (1) and formulate the model solely in force variables, as in [4]. Starting with a stable configuration

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with net force $F \equiv \sqrt{F^{x^2} + F^{y^2}} < F_s$ for all the blocks, we drive the system by gradually increasing K [17] until one of the blocks becomes unstable, i.e., K is increased to $KF_s/F_{\max}$ during loading, where $F_{\max}$ denotes the spatial maximum of F in the stable configuration. As in [4], the block is assumed to slip to its equilibrium position defined by $F = 0$, ignoring the overshot,

$$\begin{aligned} F_{i,j}^x \to 0, \quad F_{i,j}^y \to 0; \quad F_{i\pm1,j}^x \to F_{i\pm1,j}^x + \alpha F_{i,j}^x, \\ F_{i\pm1,j}^y \to F_{i\pm1,j}^y + s\alpha F_{i,j}^y; \qquad (2) \\ F_{i,j\pm1}^x \to F_{i,j\pm1}^x + s\alpha F_{i,j}^x, \quad F_{i,j\pm1}^y \to F_{i,j\pm1}^y + \alpha F_{i,j}^y, \end{aligned}$$

where $\alpha = 1/(2 + 2s)$. This locally conserves the signed force components. The resulting modification of the stress environment may trigger further slips in neighboring blocks, and hence an avalanche, until $F < F_s$ is restored for all blocks. Then K is increased again and the slip process continues.

If F_s was zero, the only stable (minimal energy) configuration would be that the blocks were exactly at the nodes of a perfect square lattice of mesh size a. The nonvanishing friction thus creates a large ensemble of coexisting metastable states which is responsible for the nontrivial dynamics. For $F_s \neq 0$ and $s \neq 0$, the toppling rules (2) do not put the blocks in their minimum energy configuration due to the couplings to their four neighbors. This ensures that a block will go on becoming unstable *ad infinitum* as long as K is increased indefinitely.

The *multiplicative* loading is motivated by the stiffening of an overlayer caused by desiccation [18], originally used to study cracks [19]. It differs from the usual *additive* loading in sandpiles [3] and stick-slip models [4–6,15], where physically the driving force arises from dropping grains onto a pile or pulling a frictional surface. Those systems are known to exhibit SOC in nonequilibrium steady states. In contrast, our system approaches an *equilibrium* instead of a genuine steady state. We stress that the dynamics does not conserve the net force F, as $F \to FF_s/F_{\max} > F$ during loadings, and it is decreased during slippings, as can be proved in the scalar limit $F_y \equiv 0$. What distinguishes ours from others is therefore the coupling of the dynamics to local conservations of the force components. Such a coupling turns out to be crucial to the critical properties.

Without loss of generality, we hereafter set $a = 1 = F_s$. Of the remaining dimensionless parameters $\{s, A, K\}$, s determines the equilibrium length scale and the dynamics through (2), A characterizes the initial disorders but is irrelevant for the equilibrium state, and K defines the "time" $t \equiv K$.

Results.—We have simulated the model for $0 \leq s \leq 1$, $A = 0.02, 0.18, 0.4$, and system sizes $20 \leq L \leq 300$. We are mainly interested in the possibility of SOC with PBC as this most differentiates us from previous works. Unless stated otherwise, PBC is assumed hereafter. To investigate the effects of spatial inhomogeneities, we also use free boundary conditions (FBC) with no block beyond the edges, and cylindrical boundary conditions (CBC) with one pair of parallel edges periodic and the other pair free. All of these boundary conditions respect the local conservations.

We study the evolution of the system by monitoring the stress field $\sigma(\vec{r}, t)$, approximated by the averaged tension of the four springs attached to a block. Using its Fourier transform $\tilde{\sigma}$, we compute the structure factor $S(\vec{k}, t) = \overline{\langle|\tilde{\sigma}(\vec{k}, t)|^2\rangle/L^2 - L^2\delta_{\vec{k},0}\langle\tilde{\sigma}(\vec{k}, t)\rangle^2}$ and the circular average $S_{\mathrm{cir}}(k, t) = \sum_{\vec{k}\in\{|\vec{k}|=k\}} S(\vec{k}, t)$, where the overline and the angular brackets mean a spatial and an ensemble average, respectively. Then $R(t) \equiv 2\pi[\sum_k k^2 S_{\mathrm{cir}}(k, t)/\sum_k S_{\mathrm{cir}}(k, t)]^{-1/2}$ measures the characteristic length scale in the stress field.

The system evolves under the influence of conservation laws from large disorders to small disorders when the blocks converge onto a perfect lattice. This is analogous to spinodal decomposition (in model C to be precise) [20] and suggests a coarsening in the stress field. We indeed find a power-law growth $R(t) \sim t^{\phi}$, where $\phi = 0.33 \pm 0.01$ (cf. [19]) for all s and A, as illustrated in Fig. 1. This universal behavior and the value of ϕ agree with spinodal decomposition.

Now we present the evidence showing the system in the long-time limit is stationary and critical in the variables relevant to the dynamics. Figure 1 illustrates that $R(t)$ is stationary as it reaches a plateau after transient. While the saturation rate depends on s and A, the value of the plateau R_0 depends only on s due to sL being the length the system has to contract to reach a stress-free state. But this cannot be achieved due to PBC, so that the blocks wind up in a frustrated state correlated over this distance. For fixed L, we verify that $R_0(s, L)$ increases linearly in s, except very close to $s = 0$ or 1. The plateau extends up to ten decades in t, until reaching the limit of numerical accuracy (i.e., 10^{-15} in F, using double precision). Further evidence comes from $\bar{F}(t)/F_s$ which measures the effective "distance" from the instability limit of the system. Its stationary fluctuations about a finite constant (Fig. 2) is an

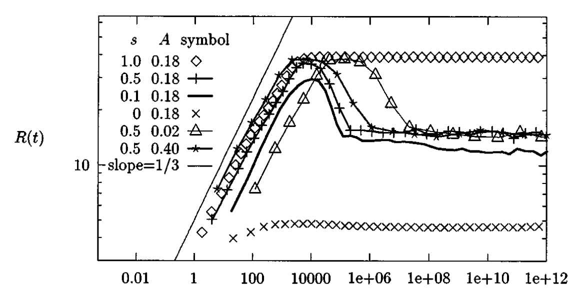

FIG. 1. Characteristic length $R(t)$ vs time $t \equiv K$ for different sets of s and A, showing the universal $1/3$ power law and plateau. $L = 40$ with PBC.

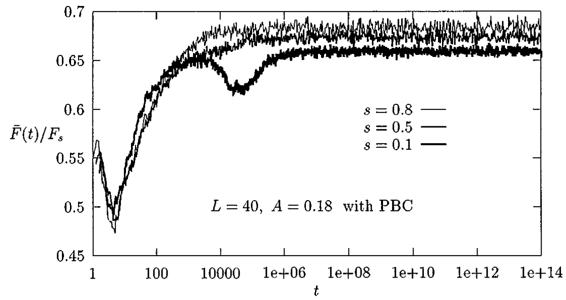

FIG. 2. Spatial averaged force on a block normalized by the threshold vs time, showing approaches to stationarity.

important characteristic of a dynamical steady state in SOC models [3–7].

Next, we show that the system is critical. First, for fixed s, we find $R_0(s, L) \propto L$, implying long-range correlations in the stress field. More importantly, R satisfies finite-size scaling $R(t, L) = t^{\phi} \tilde{R}(t/L^{1/\phi})$ (see Fig. 3), with the asymptotic behavior $\tilde{R}(x \to 0) = \text{const}$ and $\tilde{R}(x \to \infty) \propto x^{-\phi}$. This is a clear signature of criticality.

Second, the avalanches are characterized by power-law distributions, another hallmark of criticality. From (1), the force drop in one block slip is given by $F^{\text{slip}} = Ku/\alpha$ [4,16], where u denotes the slip distance. Since $F^{\text{slip}} \gtrsim F_s$, u diminishes as $1/K$ when the blocks gradually converge to a regular lattice. An avalanche consists of a correlated sequence of S block slips. Analogous to seismology [21], the "seismic moment" M and the "radiated seismic energy" E can then be computed,

$$M = K \sum_{l=1}^{S} u_l = \alpha \sum_{l=1}^{S} F_l^{\text{slip}} \gtrsim \alpha F_s S\,, \qquad (3)$$

$$E = \frac{1}{2} \sum_{l=1}^{S} F_l^{\text{slip}} u_l = \frac{\alpha}{2K} \sum_{l=1}^{S} (F_l^{\text{slip}})^2 \gtrsim \frac{F_s M}{2K}\,. \qquad (4)$$

These can be measured experimentally, as is precisely done for earthquakes. In our context, E equals the energy dissipated by friction. We find that the distribution of S approaches a power law $P(S) \sim S^{-(B+1)}$ with $B = 0.15 \pm 0.05$ (see Fig. 4), when $R(t, L) \to R_0(L)$ for $t \gtrsim L^{1/\phi}$. Consequently, both distributions of M and the scaled energy EK are stationary and follow the same power law. From these results, we conclude that our system approaches in a punctuated manner without ever reaching its final equilibrium, and is driven into a marginally stable critical state.

We also examine the effects of boundary conditions. The same exponent B within numerical uncertainty is found for PBC, FBC, and CBC. This is remarkable in view of their approach to different equilibrium states, and the usual sensitive dependence of SOC systems on boundary conditions.

The above results suggest a mechanism distinct from all previous SOC models [3–7] constructed with threshold dynamics. Those models do not exhibit SOC with PBC because open boundaries are required either (i) to balance the influx of the conserved quantity (the total number of grains or total force on the blocks), or (ii) to provide a means of desynchronization. While it is clear, due to separation of toppling and conservation, that our model is different from those associated with (i) [3], the major question remains whether the coarsening is a manifestation of a different mechanism compared to invasion from inhomogeneities in case (ii) [4,5]. To settle this, we introduce dissipation by adding $-(x, y)\kappa K$ to (F_x, F_y) in (1), which represent harmonic couplings to a rigid surface in earthquake models [4,5,15]. Since now $\alpha = 1/(2 + 2s + \kappa)$ [4,16] by (2), the dissipation in F_x and F_y are $\propto \alpha\kappa$. If the mechanism is the same as in [4,5], we would expect SOC to survive. Conversely, it is different. A decisive test is to consider the case $F_y \equiv 0$, $s = 1$, $\kappa > 0$ with FBC, in which the difference between our model and [4,5] reduces to just the way of driving. Figure 5 clearly displays the absence of criticality, strongly supporting a new mechanism is at work. Generally, the correlation length R_0 and the cutoff in $P(S)$ are found to diverge as $\kappa \to 0$, which happens to be the generic, physical limit in the context of desiccation processes [18].

In conclusion, we have studied a system which shows SOC without relying on desychronization initiated by inhomogeneities [5,6]. It reveals a coarsening mechanism whereby correlations (or "self-organized" regions) gradually build up *in the bulk*. While the associated power-law exponents are extremely robust (universal) with respect to

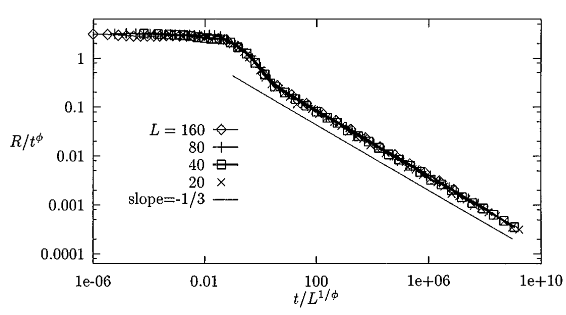

FIG. 3. Finite-size scaling plot of the characteristic length in the stress field, $R(t, L)$. $s = 0.5$, $A = 0.18$ with PBC.

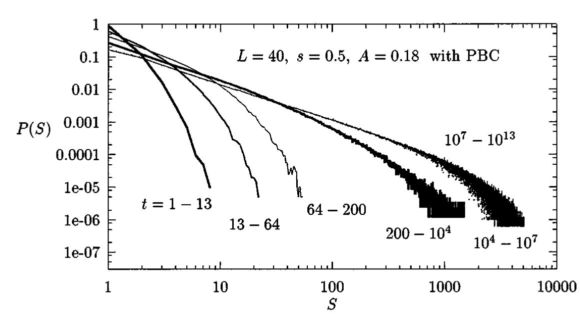

FIG. 4. $P(S)$ obtained from successive intervals in t to show the approach to power law as $R(t) \to R_0$.

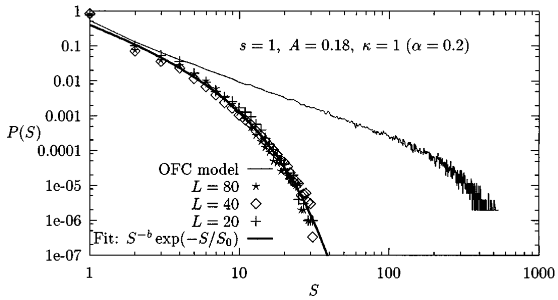

FIG. 5. $P(S)$ for $F_y \equiv 0$, $s = 1$, $\kappa = 1$ with FBC. $b = 1.5$, $S_0 = 3.9$. Power law for the OFC model (i.e., the model defined in [4]) with $\alpha = 0.2$, $L = 40$ is shown on the right for comparison.

initial disorders and strain, and inhomogeneities, conservation laws which constrain the system on the Σ_0 surface in phase space, defined by $\sum_{i,j} \vec{F}_{i,j} = 0$, are necessary for the growth of correlations. Infinitesimal dissipation *and* deviation from Σ_0 can drive the system away from Σ_0 and render it noncritical. The detailed properties in the neighborhood of Σ_0 are being pursued.

The coexistence of stationarity and approach to equilibrium in this model teaches us that transiently driven, experimental systems might, in fact, be stationary in the variable relevant to the dynamics, especially when converging to a fundamental equilibrium state. Search schemes and optimization techniques using the sweeping of a control parameter, such as in simulated annealing to access the fundamental state or optimal solution, might exhibit this kind of phenomenon characterized by a wide distribution of jumps in relaxations. Finally, the role of the multiplicative driving may be related to the production of power laws by multiplicative noise (amplification by multiplication followed by reinjection) [22].

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