CRITICAL BEHAVIOR IN SURFACE FRACTURE INDUCED BY A FRICTIONAL SUBSTRATE

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I. INTRODUCTION

Understanding the origin of the strength of materials and how they break is not only important to engineers, but also interesting and challenging for physicists. Due to the non-equilibrium nature of the processes and the importance of disorders, this understanding remains largely incomplete today, despite a long history of efforts since the early ideas by Griffith [1]. Statistical physicists get interested in these problems rather recently because suitable tools and concepts only become available over the past decade or so. Recently, some fruitful results have been obtained by applying concepts such as phase transitions, percolation and fractal to fracture [2,3], as well as by careful experiments and extensive simulations [3].

Here we consider the collective behavior in the fracture of brittle materials induced by mechanical or thermal loading. Specifically, we focus on the surface fracture phenemena of an overlayer coupled to a frictional substrate. These phenomena are ubiquitous, being abundantly present in processes of artificial or natural origin [4]. The resulting cracks span a surprisingly wide range of length scales and form intricate patterns. Physically, the tendency of the system to lower its energy (entropy playing a minor role) causes the relief of accumulated stresses either by means of slipping at the interface of contact (such as peeling off of paint on walls due to aging), or by cracking the overlayer directly (such as cracking on the surface of dried lakebeds and roads). Our goal is to study the resulting geometrical and statistical properties of cracks in the latter kind of rupture. We propose simple computational and analytic models and analyse them with concepts and methods taken from modern statistical physics.

II. MODEL

The complexity of the stress concentration and relaxation and the interaction among many cracks make the problem very hard to tackle analytically. We thus introduce a simple spring-block model based on the competition between stick-slip mechanism and cracking [5]. It consists of an array of interconnected blocks and springs, in contact with a frictional substrate. The array models the elastic response of the overlayer in a coarse-grained manner, while its frictional interaction with the substrate is modeled by a threshold condition. The threshold condition specifies that a block will slip to a new equilibrium position if the net force F acting on it from its neighbors exceeds a given threshold F_s. Moreover, if the tension b on a spring exceeds a breaking threshold F_c, the spring breaks and a microcrack is born. Stress amplification at the crack tip may then trigger the microcrack to grow to macroscopic size and ultimately split the sample into pieces.

Denoting the spring coefficient by K, relaxed spring length by l the lattice constant by a, we find that to linear order in displacements (x, y) the force acting on a block at site (i, j) is given by [6,5]

$$\begin{aligned} F_x &= (a-l+x_E-x)K_E + (x_N-x)sK_N + \\ &\quad (-a+l+x_W-x)K_W + (x_S-x)sK_S \\ F_y &= (y_E-y)sK_E + (a-l+y_N-y)K_N + \\ &\quad (y_W-y)sK_W + (-a+l+y_S-y)K_S, \end{aligned} \tag{1}$$

where the subscripts "ESWN" refer to the $(i\pm1, j\pm1)$ directions. $s = 1 - l/a > 0$ is the internal tensile strain. Similar linear relationship holds for the spring tension b.

This model has two basic elements: The stress loading mechanism and the modes of stress relief. There are many possible ways to load the system, here we only discuss the simplest case [7] modeled after desiccation experiments [8]. Starting with small K, we impose stress by slowly increasing K. As a result, all F on the blocks and b of the springs increase proportionately. For the stress relief, the basic idea is that when the material is increasingly strained, it seeks to relieve at the weakest point, either within the layer (cracking) or at the interface (slipping). The relative dominance of one mode over another, embodied in the ratio $\kappa \equiv F_c/F_s$, dictates the wide range of static and dynamical properties of fracture.

Using Eq. (1), it can be shown that a slip event causes the following redistribution of the forces among the nearest neighbors [6,5]:

$$\begin{aligned} &F^x \to 0, \quad F^y \to 0; \\ &F^x_E \to F^x_E + \alpha F^x, \quad F^y_E \to F^y_E + s\alpha F^y; \\ &F^x_W \to F^x_W + \alpha F^x, \quad F^y_W \to F^y_W + s\alpha F^y; \\ &F^x_N \to F^x_N + s\alpha F^x, \quad F^y_N \to F^y_N + \alpha F^y; \\ &F^x_S \to F^x_S + s\alpha F^x, \quad F^y_S \to F^y_S + \alpha F^y; \end{aligned} \tag{2}$$

where $\alpha = 1/(2+2s)$ specifies the amount of redistribution. Notice that $\sum_{i,j} \vec{F}_{i,j}$ is unchanged by a slip, hence the dynamics is naturally constrained by *conservation* of the force components. For the spring-breaking event, we simply put $K = 0$ thereafter for the given spring, and then check for possible slippings in the connected blocks. We assume the system evolves in the *quasi-static limit* [9], where slipping and cracking take place instantaneously during an increase of K. This is an excellent approximation of real situations such as in [8], where the time scale in the driving is usually much longer than that of stress relief. The advantage is that the dissipated dynamics arising from friction *during* slippings do not need to be treated explicitly [10]. The model as defined by Eq. (1), (2) and cracking events under the quasi-static limit is then reduced to a coupled-map lattice which can be updated efficiently on a computer.

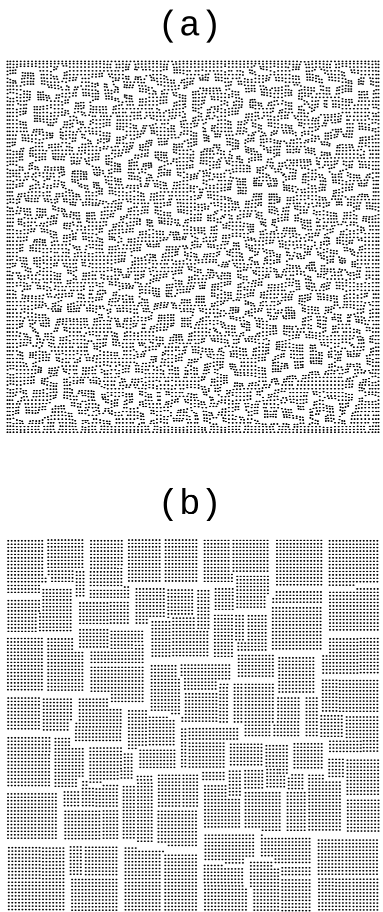

FIG. 1. Typical patterns for $L = 100$, for $\{s, \kappa\}$ equal to (a) $\{0.5, 1.6\}$, large internal strain with a weak strength; and (b) $\{0.1, 6\}$, small strain with a moderate strength.

III. RESULT

First we need to identify the control parameters. By choosing the units of length, force and time, we may set $a = 1 = F_s$ (hence $l = 1 - s$, $F_c = \kappa$), and time $t \equiv K$, since K increases monotonically. This leaves us with a minimal set $\{s, \kappa\}$, which defines our parameter space on which the phase diagram may be drawn. An immediate implication is that a strong overlayer (large F_c) is equivalent to having a weak substrate coupling (small F_s), when physical quantities are expressed in the present units. For simplicity, from now on we will interpret κ as material strength.

Fig. 1 displays two snapshots of the crack pattern for two different values of $\{s, \kappa\}$, corresponding to the case of a large strain with a weak material in Fig. 1(a), and a small strain with moderate strength in Fig. 1(b). Several qualitative differences are evident: The cracks in Fig. 1(a) are more tenuous and isolated than in Fig. 1(b). Less stress amplification at crack tips due to smaller radius of curvature results in shorter cracks and a more diffused network. These are consistent with the fact that a smaller κ also corresponds to a stronger substrate coupling F_s, hence more limitation of stress amplification, prohibiting crack propagation.

To quantify the different behaviors, we introduce the crack size c as the number of consecutive broken springs in a given spring-breaking event, and compute the cumulative distribution function $D_>(c)$ for having a crack size greater than or equal to c. On the $\{s, \kappa\}$-plane, while the system shows little difference for different κ at fixed, small strain s, it exhibits a striking phase transition at large, fixed s when κ is varied. There are two regimes [5]:

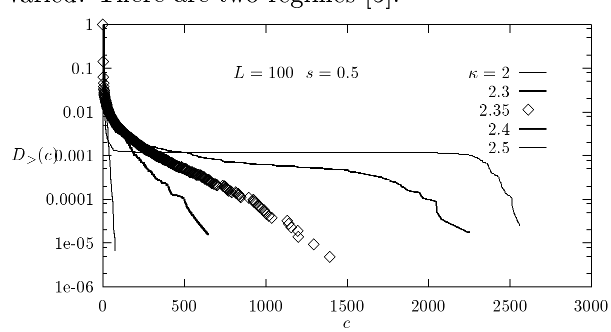

FIG. 2. Cumulative distribution of crack size per event. Note short (long)-range correlation below (above) $\kappa^* \approx 2.35$.

1. Isolated cracks— This is the case when κ is small, as in Fig. 1(a). $D_>(c)$ decays exponentially (see Fig. 2). It implies short-range spatial correlation in the cracks. This result can be understood on the basis of the effect of κ on the growth of correlation in the stress field $\sigma(\vec{r}, t)$. Initially, the correlation is very short-range due to initial randomness in the

positions of the blocks. Correlation length is $R = O(1)$. Because these randomness is annealed, it coarsens and R grows as a result of slippings. According to Eq. (2), slippings distribute $\vec{F}$ conservatively and hence build up spatial correlations in the forces or the stress field, in analogy to domain growth in binary mixture via spinodal decomposition [11], The stress field here plays the same role as the concentration field in binary mixture. However, a small $\kappa = F_c$ means that cracks soon appear [12], before the domains have a chance to grow to large sizes. The cracks in turn cannot propagate very far because there are still small length-scale randomness or fluctuations in the stress field.

2. Correlated cracks— The opposite is true for large $\kappa > \kappa^*$. Here the slippings dominate and R manages to grow up to the system size L. When the cracks finally appear within this highly correlated stressed environment, they propagate far to break the sample in a violent catastrophe. Consequently, $D_>(c)$ shows long-range order (see Fig. 2). Intuitively, we can understand why we have catastrophic cracks for large κ. That is because large κ correponds to a small F_s, and a small F_s encourages slippings to occur, which in turn amplifies stress concentration at crack tips. Thus, given a long-range correlated stress field (cf. [12]), large stress amplifications drive the system to an instability. In the limit $F_s \to 0$, the layer simply decouples from the substrate and we recover the Griffith case [1], where the stress fields are indeed long-range, decaying by power laws.

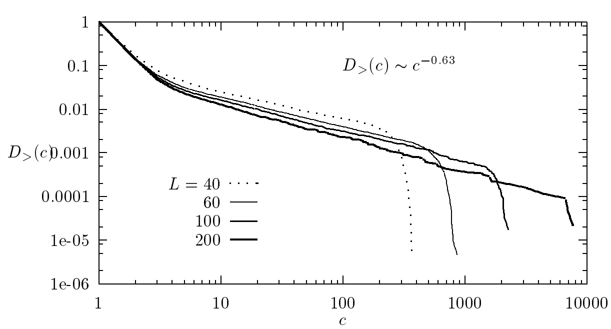

FIG. 3. Cumulative distribution of crack size per event showing power-law decay $D_>(c) \sim c^{-0.63}$ at $\kappa = \kappa^*$.

The transition is reminiscent of ordinary phase transitions in thermodynamic systems [13]. The main question is whether our transition is first or continuous order. We find that there exists an s-dependent critical value κ^* at which the correlation length diverges, manifesting in a power-law decay of $D_>(c)$. This is shown in Fig. 3. The physical meaning of this power law is that the network of cracks *per event* reaches a percolating cluster as $\kappa \to \kappa^*$ from below. This is illustrated by the snapshots in Fig. 4. Hence we conclude that the transition is continuous. The exponent η assiciated with $D_>(c) \sim c^{-\eta}$ appears to be nonuniversal. It varies from 0.75 ± 0.05 to 0.63 ± 0.02 as s changes from 0.8 to 0.5. Another signature of the phase transition appears in the distribution $P(f)$ of fragment sizes f [5]. For $s < 0.4$, the qualitative changes seem to disappear, so that there could be only one phase for all κ. The physical reason for this disappearance can be traced to the dependence of stress fluctuations on s [5]: smaller s has larger fluctuations, hence suppressing the above effect of correlation on crack propagation.

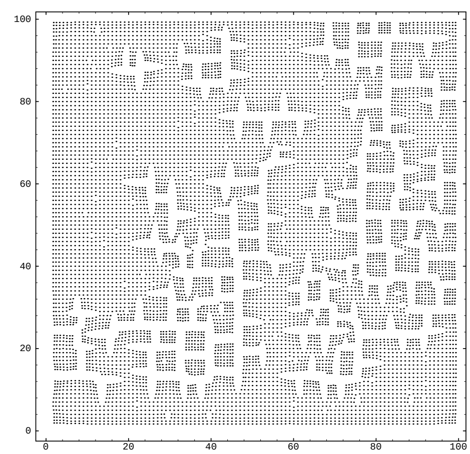

FIG. 4. Crack pattern for $L = 100$ at $s = 0.5$, $\kappa = 2.3$, near transition point $\kappa^* = 2.35$, showing almost percolating cracks.

It is interesting to note that similar transitions have been observed in the functional form of $P(f)$ in fragmentation experiments [14]. However, the resemblence may be superficial as the nature of the transition may well depend sensitively on the loading scheme, which is usually extremely fast in fragmentations (such as by impact or explosion).

In a related work, the effect of disorder in the Griffith limit (i.e., without stress transfer limitation among the elements, such as the blocks above) is studied analytically by means of an idealized model consisting of a bundle of parallel fibers [15]. In that model, when one fiber breaks, the relieved stress is equally transferred to the remaining fibers. Disorder is introduced in the form of a broad distribution of breaking thresholds F_c for the fibers. With sufficient disorder and the existence of a minimum F_c, we discovered a tri-critical rupture behavior, separating in phase space a first-order and a continuous regime [15]. Although it is a mean-field model, the result is consistent with simulations using the more realistic spring-block model. This finding explains why some systems exhibit clearer precursors before global rupture than others, and hence is useful for failure prediction analyses.

IV. CONCLUSION

In summary, we have achieved our initial goal of formulating a simple yet realistic model of surface fracture, characterized by a minimal set of control parameters. A novel phase transition is found to be induced by the substrate coupling. The simplicity of the model allow us to understand the surface fracture phenomena in a physical and intuitive way. It demonstrates that the model is suitable for investigating the properties of cracks and their propagation.

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