Appendix

Proof of Lemma 1:

First, if y_s is the stock after trade in a steady state, then $P(y_s) = f(y_s)$. The reason is that in the steady state we have $t((1-\mu)y_s) = y_s$. Hence, the consumer's arbitrage equation $f(y_s) - P(y_s) = \rho[f(y_s) - P(t((1-\mu)y_s))]$ implies $P(y_s) = f(y_s)$.

To prove that a steady state must exist, we first show that if P and f cross, then there always is a steady state. Then we establish that P and f necessarily cross.

We claim that if $S = \{q : P(q) = f(q)\}$ is non-empty then $q' = \max S$ is a steady state. To see this, observe that q' is the maximum state to have acceptance price $P(q')$. Indeed, if there existed $q'' > q'$ with $P(q'') = P(q')$, then from the definition of q' we must have $f(q'') < f(q')$. Furthermore, since $t(\cdot)$ is non-decreasing, and since $P(\cdot)$ is non-increasing, we have $P(t((1-\mu)q'')) \leq P(t((1-\mu)q'))$. This implies the contradiction $P(q'') = (1-\rho)f(q'') + \rho P(t((1-\mu)q'')) < (1-\rho)f(q') + \rho P(t((1-\mu)q')) = P(q')$. Hence following the offer $P(q')$, all $q \leq q'$ accept and all $q > q'$ reject. Now from $P(q') = (1-\rho)f(q') + \rho P(t((1-\mu)q'))$ and $P(q') = f(q')$ we have $P(t((1-\mu)q'))$ $= P(q')$. Thus when the state before trade is $(1-\mu)q'$ the monopolist's price $P(t((1-\mu)q'))$ $= P(q')$ leads to a state after trade equal to q', i.e. $t((1-\mu)q') = q'$.

Suppose now that there is a stationary equilibrium which does not have any steady state. We claim that this implies $P(q) < \overline{v}$ for $q \in [0, \hat{q}]$ and $P(q) > \underline{v}$ for $q \in (\hat{q}, 1]$. To see this, note that since there is no steady state, it follows from the previous paragraph that the set S is empty, i.e. $P(q) \neq \overline{v}$ for any $q \in [0, \hat{q}]$ and $P(q) \neq \underline{v}$ for any $q \in (\hat{q}, 1]$. An argument similar to Fudenberg, Levine and Tirole (1985, Lemma 2) establishes that $P(q) \geq \underline{v}$ for all $q \in [0, 1]$. Hence we necessarily have $P(q) > \underline{v}$ for all $q \in (\hat{q}, 1]$. Furthermore, we cannot have $P(q) > \overline{v}$ for some $q \in [0, \hat{q}]$. Otherwise, since $P(\cdot)$ is non-increasing, we would have $P(0) > \overline{v}$. But then $P(0) = (1-\rho)\overline{v} + \rho P(t(0))$ implies $P(t(0)) > P(0)$. This is a contradiction, as $t(0) \geq 0$ and $P(\cdot)$ is a non-increasing function.

Next, we show that $P(q) < \overline{v}$ for $q \in [0, \hat{q}]$ and $P(q) > \underline{v}$ for $q \in (\hat{q}, 1]$ imply that the total stock is increasing for $q \in [0, \hat{q}]$ and decreasing for $q \in (\hat{q}, 1]$, and that this yields a contradiction.

By the consumer's arbitrage equation, we have $\overline{v} - P(q) = \rho[\overline{v} - P(t((1-\mu)q)]$ for all $q \in [0, \hat{q}]$. Since $\rho \in (0, 1)$ for all $z > 0$ and $\overline{v} - P(q) > 0$, we have $\overline{v} - P(q) < \overline{v} - P(t(1-\mu)q)$ which implies $P(t((1-\mu)q)) < P(q)$ for all $q \in [0, \hat{q}]$. Since $P(\cdot)$ is decreasing, we have $t((1-\mu)q) > q$ for all $q \in [0, \hat{q}]$.

A similar argument also establishes that $t((1-\mu)(\hat{q} + \epsilon)) < \hat{q} + \epsilon$ for all $\epsilon \in (0, 1-\hat{q}]$. Hence,

$\lim_{\epsilon \to 0} t((1-\mu)(\hat{q}+\epsilon)) \leq \hat{q} < t((1-\mu)\hat{q})$. Since $T(\cdot)$ is upper hemi-continuous, $\lim_{\epsilon \to 0} t((1-\mu)(\hat{q}+\epsilon)) \in T((1-\mu)\hat{q})$. This contradicts the definition of $t(\cdot) = \min T(\cdot)$. *Q.E.D.*

Lemma 4 *If $f(q) = f(q')$ and $P(q) = P(q')$ for some $q < q'$ then q cannot be a steady state.*

Proof : If q were a steady state, then when the state after depreciation is $(1-\mu)q$ the monopolist will charge $P(q)$. All buyers in $((1-\mu)q, q']$ will accept this, contradicting that $t((1-\mu)q) = q$. *Q.E.D.*

Proof of Lemma 2:

First, we show that $y \in (\hat{q}, 1)$ cannot be a steady state. Suppose not; then Lemma 1 implies that $P(y) = \underline{v}$. Since $P(\cdot)$ is a decreasing function, we have $P(q) \leq \underline{v}$ for $q \geq y$. Similar to Fudenberg, Levine, and Tirole (1985), we have $P(q) \geq \underline{v}$ for all $q \in [0, 1]$. Hence, $P(q) = \underline{v}$ for all $q \in [y, 1]$. Lemma 4 then shows that y cannot be a steady state.

To prove (ii), suppose that $\hat{q}$ is a steady state so that $P(\hat{q}) = \overline{v}$. Then since $P(\cdot)$ is decreasing, and since (as shown in the proof of Lemma 1) $P(y) \leq \overline{v}$ for $y \leq \hat{q}$, we have $P(y) = \overline{v}$ for all $y \in [0, \hat{q}]$. Hence Lemma 4 shows that $y < \hat{q}$ cannot be a steady state. We conclude that the only possible steady state other than $\hat{q}$ is 1.

To prove (iii) suppose that $0 < y^* < \hat{q}$ is a steady state. Then we must have $P(y) = \overline{v}$ for all $y \in [0, y^*]$, so Lemma 4 implies that no $y < y^*$ can be a steady state, and that $P(q) < \overline{v}$ for $q \in (y^*, \hat{q}]$ (otherwise y^* could not be a steady state). Thus by Lemma 1 no $y \in (y^*, \hat{q}]$ can be a steady state, leaving $y = 1$ as the only other possible steady state. Suppose $y = 1$ were not a steady state, so $P(1) > \underline{v}$. This would imply $P(q) > \underline{v}$ for all $q \in (\hat{q}, 1]$. We can then use the argument at the end of the proof of Lemma 1 to obtain a contradiction.

To prove (i), note that if there is no steady state in $[0, \hat{q}]$, then by the first paragraph $y = 1$ is the only possible steady state. It then follows from Lemma 1 that $y = 1$ must be a steady state. *Q.E.D.*

Proof of Theorem 1:

We start by defining $\bar{\mu}$. To this effect, let $\bar{x}_0 = 1 - \mu$, $\bar{x}_1 = (1-\mu)\hat{q}$, $\bar{x}_2 = \frac{\hat{q}\overline{v} - \underline{v}}{\overline{v} - \underline{v}}$, and for $k \geq 3$ let

$$\bar{x}_k = (1-\mu)^{-1}\Big(\bar{x}_{k-1} - (\bar{x}_{k-2} - \bar{x}_{k-1})\frac{\overline{v}}{\rho^{k-2}(\overline{v} - \underline{v})}\Big). \tag{A-1}$$

Let $\Lambda = \{\mu \geq 0 \mid \exists m < \infty \text{ s.t. } \bar{x}_{m+1} < 0 \leq \bar{x}_m \text{ and s.t. } \{\bar{x}_k\}_{i=0}^{m} \text{ is a strictly decreasing sequence}\}$, and define $\bar{\mu} = \sup \Lambda$. Note that $0 \in \Lambda$ and that Λ is open in $\mathbb{R}_+$. It follows that $\bar{\mu} > 0$. Now let

μ_2 be the solution to $(1-\mu)\hat{q} = \frac{\hat{q}\overline{v}-\underline{v}}{\overline{v}-\underline{v}}$. Then for $\mu \geq \mu_2$ we have $\bar{x}_2 \geq \bar{x}_1$, so $\bar{\mu} \leq \mu_2 < 1$.

First, we show that if $\mu' \in \Lambda$ and $\mu < \mu'$ then $\mu \in \Lambda$, i.e. $\Lambda = [0, \bar{\mu})$. We will prove this by showing that $\Delta_k = (\bar{x}_{k-1} - \bar{x}_k)$ is decreasing in μ. By definition we have $\bar{x}_0 = 1-\mu$, $\bar{x}_1 = (1-\mu)\hat{q}$, $\bar{x}_2 = \frac{\hat{q}\overline{v}-\underline{v}}{\overline{v}-\underline{v}}$, and

$$\bar{x}_k = (1-\mu)^{-1}\Big(\bar{x}_{k-1} - a_{k-2}\Delta_{k-1}\Big),$$

where $a_k = \frac{\overline{v}}{\rho^k(\overline{v}-\underline{v})}$. Thus $\frac{d\Delta_k}{d\mu} < 0$ for $k \leq 2$ and

$$\begin{aligned}\frac{d\bar{x}_3}{d\mu} &= (1-\mu)^{-2}(a_1\bar{x}_1 + \bar{x}_2 - 2a_1\Delta_2)\\ &= (1-\mu)^{-2}(a_1\bar{x}_1 - \bar{x}_2 + 2(1-\mu)\bar{x}_3).\end{aligned}$$

Thus $\frac{d\bar{x}_3}{d\mu} > 0$ and $\frac{d\Delta_3}{d\mu} < -2(1-\mu)^{-1}\bar{x}_3$. We now prove by induction that if $\frac{d\bar{x}_{k-1}}{d\mu} > 0$ and $\frac{d\Delta_{k-1}}{d\mu} < -(k-2)(1-\mu)^{-1}\bar{x}_{k-1}$ then $\frac{d\bar{x}_k}{d\mu} > 0$ and $\frac{d\Delta_k}{d\mu} < -(k-1)(1-\mu)^{-1}\bar{x}_k$. Indeed, $\Delta_k = (1-(1-\mu)^{-1})\bar{x}_{k-1} + (1-\mu)^{-1}a_{k-2}\Delta_{k-1}$, so

$$\begin{aligned}\frac{d\Delta_k}{d\mu} &< -(1-\mu)^{-2}\bar{x}_{k-1} + (k-1)(1-\mu)^{-2}a_{k-2}\Delta_{k-1} + (1-\mu)^{-1}a_{k-2}\frac{d\Delta_{k-1}}{d\mu}\\ &\leq -(1-\mu)^{-2}\Big(\bar{x}_{k-1} - (k-1)a_{k-2}\Delta_{k-1} + (k-2)a_{k-2}\bar{x}_{k-1}\Big)\\ &< -(1-\mu)^{-2}(k-1)\Big(\bar{x}_{k-1} - a_{k-2}\Delta_{k-1}\Big)\\ &< -(k-1)(1-\mu)^{-1}\bar{x}_k,\end{aligned}$$

where the third inequality follows from $a_{k-2} > 1$.

Next, we show that for $\mu \in \Lambda$ there exists a stationary equilibrium with unique steady state $y_s = 1$. For $\mu \in \Lambda$ define the triplet $\{P(\cdot), R(\cdot), t(\cdot)\}$ as follows.

$$P(q) = \begin{cases} p_m & \text{for } q \in [0, \bar{y}_m] \\ p_k & \text{for } q \in (\bar{y}_{k+1}, \bar{y}_k] \text{ and } k \in \{m-1, ..., 0\} \end{cases} \tag{A-2}$$

$$t(q) = \begin{cases} \bar{y}_{m-1} & \text{for } q \in [0, \bar{x}_m], \\ \bar{y}_{k-1} & \text{for } q \in (\bar{x}_{k+1}, \bar{x}_k], k \in \{m-1, \ldots, 2\} \\ 1 & \text{for } q \in (\bar{x}_2, 1-\mu] \end{cases} \tag{A-3}$$

$$R(q) = \begin{cases} (\bar{y}_{m-1} - q)P(\bar{y}_{m-1}) + \delta R(\bar{x}_{m-1}) & \text{for } q \in [0, \bar{x}_m], \\ (\bar{y}_{k-1} - q)P(\bar{y}_{k-1}) + \delta R(\bar{x}_{k-1}) & \text{for } q \in (\bar{x}_{k+1}, \bar{x}_k], k \in \{m-1, \ldots, 2\} \\ (1-q)\underline{v} + \frac{\delta\mu\underline{v}}{1-\delta} & \text{for } q \in (\bar{x}_2, 1-\mu] \end{cases} \tag{A-4}$$

where $p_k = \overline{v} - \rho^k(\overline{v} - \underline{v})$ and $\bar{y}_k = \frac{\bar{x}_k}{1-\mu}$, for $k = 0, ..., m$. The sequence $\{\bar{x}_k\}_{k=2}^{m+1}$ has been defined so that when the state is $\bar{x}_k$ the monopolist is indifferent between selecting $\bar{y}_{k-1}$ and $\bar{y}_{k-2}$, i.e.

defining $\pi(y;x) = P(y)(y-x) + \delta\, R((1-\mu)y)$ we have $\pi(\bar{y}_{k-1};\bar{x}_k) = \pi(\bar{y}_{k-2};\bar{x}_k)$. Let $h_k(x) = \pi(\bar{y}_{k-1};x) - \pi(\bar{y}_{k-2};x)$; then $h_k(\bar{x}_k) = 0$ and $\frac{dh_k}{dx} = -P(\bar{y}_{k-1}) + P(\bar{y}_{k-2}) < 0$. It follows that for $x < \bar{x}_k$ we have $\pi(\bar{y}_{k-1};x) > \pi(\bar{y}_{k-2};x)$, and for $x > \bar{x}_k$ we have $\pi(\bar{y}_{k-1};x) < \pi(\bar{y}_{k-2};x)$. Since $\pi(y;x)$ is strictly increasing in y on any of the intervals $[0,\bar{y}_m], (\bar{y}_m,\bar{y}_{m-1}], ..., (\bar{y}_1, 1]$ it follows that that $t(x)$ is the smallest solution to (3). It is straightforward to verify that (4) is also satisfied, so $\{P(\cdot), R(\cdot), t(\cdot)\}$ is a stationary triplet for $\mu \in \Lambda$.

Finally, consider any stationary equilibrium whose unique steady state is $y_s = 1$, and let $\{P_0(\cdot), R_0(\cdot), t_0(\cdot)\}$ be the associated stationary triplet. We claim that $\{P_0(\cdot), R_0(\cdot), t_0(\cdot)\} = \{P(\cdot), R(\cdot), t(\cdot)\}$.

First, we show that $P_0(y) = P(y)$ for all $y \in (\hat{q}, 1]$. Indeed, since $\hat{q}$ is not a steady state, it must be that $P(\hat{q}) < \bar{v}$. Otherwise, we would have $P(q) = f(q) = \bar{v}$ for all $q \leq \hat{q}$. But as shown in the proof of Lemma 1, this would imply that $\hat{q}$ is a steady state. Now $P(\hat{q}) < \bar{v}$ and the existence of $\varepsilon > 0$ s.t. $P(q) > \underline{v}$ for all $q \in (\hat{q}, \hat{q}+\varepsilon)$ would yield the same contradiction as in the last two paragraphs of the proof of Lemma 1. We conclude that $P_0(q) = \underline{v}$ for all $q \in (\hat{q}, 1]$.

Next, we claim that $t_0(\bar{x}_1) = 1$. Indeed, $P(\hat{q}) < \bar{v}$ and (4) applied to $q = \hat{q}$ imply that $P_0(t_0(\bar{x}_1)) < P_0(\hat{q})$. Since $P_0(\cdot)$ is non-increasing, we therefore must have $t_0(\bar{x}_1) > \hat{q}$. Finally, since $P_0(q) = \underline{v}$ for all $q > \hat{q}$, (3) implies $t_0(\bar{x}_1) = 1$.

By the left-continuity of $t_0(\cdot)$, there exists $\varepsilon > 0$ s.t. $t_0(x) = 1$ for all $x \in (\bar{x}_1 - \varepsilon, \bar{x}_0]$. Furthermore, (4) implies $P_0(y) = \rho\bar{v} + (1-\rho)\underline{v}$ for $y \in (\bar{y}_1 - \frac{\varepsilon}{1-\mu}, \bar{y}_1]$. Now define $\bar{\bar{x}}_2 = \inf\{x : t_0(x) = 1\}$. We claim that $\bar{\bar{x}}_2 = \bar{x}_2$, so $P_0(\frac{x}{1-\mu}) = P(\frac{x}{1-\mu})$ and $t_0(x) = t(x)$ for $x \in (\bar{x}_2, \bar{x}_1]$. To see this, recall that we shown above that $\pi(\bar{y}_1;x) < \pi(\bar{y}_0,x)$ for $x > \bar{x}_2$, and $\pi(\bar{y}_1;x) > \pi(\bar{y}_0,x)$ for $x < \bar{x}_2$. Since $\pi_0(\bar{y}_1;x) = \pi(\bar{y}_1;x)$ and $\pi_0(\bar{y}_0,x) = \pi(\bar{y}_0,x)$ we must have $\bar{\bar{x}}_2 = \bar{x}_2$.

The same argument can now be applied inductively. Suppose (A-2)-(A-4) hold for all $i = 0, ..., k$. Then since we must have $P_0(\bar{y}_{k+1}) < \bar{v}$, we know that $P_0(\bar{y}_{k+1}) > P_0(t_0(\bar{x}_{k+1}))$ and so $t_0(\bar{x}_{k+1}) > \bar{y}_{k+1}$. Since $\bar{\bar{x}}_{k+1} = \inf\{x : t_0(x) = \bar{y}_{k-1}\}$ condition (A-3) implies that $t_0(\bar{x}_{k+1}) = \bar{y}_k$. Similarly to the previous paragraph, we can then use left-continuity of $t_0(\cdot)$ to define $\bar{\bar{x}}_{k+2} = \inf\{x : t_0(x) = \bar{y}_k\}$, and conclude that $\bar{\bar{x}}_{k+2} = \bar{x}_{k+2}$. *Q.E.D.*

Proof of* Corollary *1:

By the definition of $\bar{x}_k$, for $k \geq 3$ we have $\lim_{z\to 0} \bar{x}_k = \lim_{z\to 0}(\bar{x}_{k-1} - (\bar{x}_{k-2} - \bar{x}_{k-1})\frac{\bar{v}}{\bar{v}-\underline{v}})$. Hence, $\lim_{z\to 0}(\bar{x}_{k-1} - \bar{x}_k) > \lim_{z\to 0}(\bar{x}_{k-2} - \bar{x}_{k-1})$. By induction,

$$\lim_{z\to 0}(\bar{x}_{k-1} - \bar{x}_k) > \lim_{z\to 0}(\bar{x}_{k-2} - \bar{x}_{k-1}) > \hat{q} - \bar{x}_2 = \frac{(1-\hat{q})\underline{v}}{\bar{v} - \underline{v}}.$$

It follows that as z tends to zero, it takes the monopolist at most $\hat{m} \leq \frac{\overline{v}-\underline{v}}{(1-\hat{q})\underline{v}} + 1$ steps to sell to all consumers. *Q.E.D.*

Proof of Theorem 2:

Let $\tilde{x}_0 = (1-\mu)\hat{q}$ and iteratively define the sequence $\{\tilde{x}_k\}_{k=0}^{\infty}$ by $\tilde{x}_k = \frac{\tilde{p}_{k-1}\tilde{x}_{k-1}}{\mu\underline{v}+(1-\mu)\tilde{p}_{k-1}}$, where $\tilde{p}_k = (1-\rho^k)\underline{v} + \rho^k\overline{v}$. Observe that $\tilde{x}_0 < \tilde{x}_1 < \cdots$. If $x_\infty = \lim_{k\to\infty} \tilde{x}_k \geq 1-\mu$, then set $m = \sup\{k : \tilde{x}_k < 1-\mu\}$ and redefine $\tilde{x}_{m+1} = 1-\mu$. Let $\tilde{y}_k = \frac{\tilde{x}_k}{1-\mu}$, and define the triplet:

$$
\begin{aligned}
P(q) &= \begin{cases} \overline{v} & \forall q \in [0,\hat{q}] \\ \tilde{p}_k & \forall q \in (\tilde{y}_{k-1}, \tilde{y}_k] \text{ and } k = 1,\ldots,m+1 \end{cases} \\
t(q) &= \begin{cases} \hat{q} & \forall q \in [0,\tilde{x}_1] \\ \tilde{y}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2,\ldots,m+1 \end{cases} \\
R(q) &= \begin{cases} \left(\frac{\delta\mu\hat{q}}{1-\delta} + (\hat{q}-q)\right)\overline{v} & \forall q \in [0,\tilde{x}_1] \\ \left(\frac{\mu\tilde{y}_{k-1}}{1-\delta} - (q-\tilde{x}_{k-1})\right)\tilde{p}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2,\ldots,m+1 \end{cases}
\end{aligned}
\tag{A-5}
$$

If $x_\infty = \lim_{k\to\infty} \tilde{x}_k < 1-\mu$, define $\tilde{R}(x) = \left(\frac{\mu\tilde{x}_{k-1}}{(1-\delta)(1-\mu)} - (x-\tilde{x}_{k-1})\right)\tilde{p}_{k-1}$, for $x \in (\tilde{x}_{k-1}, \tilde{x}_k]$ and $k \geq 1$. Set $x^* = \max\{x \in [(1-\mu)\hat{q}, x_\infty] : \tilde{R}(x) \geq R_1(x)\}$, where $R_1(q) = \left(\frac{\delta\mu}{1-\delta} + (1-q)\right)\underline{v}$, and let m be such that $x^* \in (\tilde{x}_m, \tilde{x}_{m+1}]$. To see that m exists note that $\tilde{R}(x_\infty) - R_1(x_\infty) = (\frac{\mu x^*}{(1-\delta)(1-\mu)} - [\frac{\delta\mu}{1-\delta} + (1-x^*)])\underline{v} < 0$ whenever $x_\infty < 1-\mu$, so we have $x^* < x_\infty$. Furthermore the definition of $\underline{\mu}$ implies that $x^* \geq (1-\mu)\hat{q}$, with strict inequality when $\mu > \underline{\mu}$. Next, redefine $\tilde{x}_{m+1} = x^*$, and define the triplet:

$$
\begin{aligned}
P(q) &= \begin{cases} \overline{v} & \forall q \in [0,\hat{q}] \\ \tilde{p}_k & \forall q \in (\tilde{y}_{k-1}, \tilde{y}_k] \text{ and } k = 1,\ldots,m+1 \\ \underline{v} & \forall q \in (\tilde{y}_{m+1}, 1] \end{cases} \\
t(q) &= \begin{cases} \hat{q} & \forall q \in [0,\tilde{x}_1] \\ \tilde{y}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2,\ldots,m+1 \\ 1 & \forall q \in (\tilde{x}_{m+1}, 1-\mu] \end{cases} \\
R(q) &= \begin{cases} \left(\frac{\delta\mu\hat{q}}{1-\delta} + (\hat{q}-q)\right)\overline{v} & \forall q \in [0,\tilde{x}_1] \\ \left(\frac{\mu\tilde{y}_{k-1}}{1-\delta} - (q-\tilde{x}_{k-1})\right)\tilde{p}_{k-1} & \forall q \in (\tilde{x}_{k-1}, \tilde{x}_k] \text{ and } k = 2,\ldots,m+1 \\ \left(\frac{\delta\mu}{1-\delta} + (1-q)\right)\underline{v} & \forall q \in (\tilde{x}_{m+1}, 1-\mu] \end{cases}
\end{aligned}
\tag{A-6}
$$

Let us now show that the stationary triplets defined above satisfy (3). We will consider the case $x_\infty < 1-\mu$; the proof for the case $x_\infty \geq 1-\mu$ is similar. Note that $\pi(y;x)$ is strictly increasing in

y on any of the intervals $[0, \tilde{y}_0], (\tilde{y}_0, \tilde{y}_1], ..., (\tilde{y}_{m+1}, 1]$, so $T(x) \subset \{\tilde{y}_0, ...\tilde{y}_{m+1}, 1\}$. Furthermore, the sequence $\{\tilde{x}_k\}$ is constructed such that when the state is $\tilde{x}_k$ the monopolist is indifferent between selecting $\tilde{y}_{k-1}$ and $\tilde{y}_k$, i.e. $\pi(\tilde{y}_{k-1}; \tilde{x}_k) = \pi(\tilde{y}_k; \tilde{x}_k)$. Let $h_{m+1}(x) = \pi(1; x) - \pi(\tilde{y}_m; x)$, and for $k \leq m$ let $h_k(x) = \pi(\tilde{y}_k; x) - \pi(\tilde{y}_{k-1}; x)$. Then $h_k(\tilde{x}_k) = 0$ for all k, $\frac{dh_{m+1}}{dx} = -(\underline{v} - \tilde{p}_{k-1}) > 0$, and $\frac{dh_k}{dx} = -(\tilde{p}_k - \tilde{p}_{k-1}) > 0$ for $k \leq m$. Hence for all k we have $h_k(x) \lesseqgtr 0$ as $x \lesseqgtr \tilde{x}_k$. We conclude that $t(x)$ solves (3). It is straightforward to verify that (4) is also satisfied, so $\{P(\cdot), R(\cdot), t(\cdot)\}$ is a stationary triplet for $\mu \geq \underline{\mu}$.

Finally, let $\{P_0(\cdot), R_0(\cdot), t_0(\cdot)\}$ be the triplet associated with some stationary equilibrium having $\hat{q}$ as a steady state. We will show that $\{P_0, R_0, t_0\}$ is unique, and hence must equal $\{P, R, t\}$. Define $\breve{p}_1 = \sup_{q>\hat{q}} P_0(q)$.

Suppose first that $\breve{p}_1 = \underline{v}$. Then $P_0(q) = \overline{v}$ for $q \in (0, \hat{q}]$ and $P_0(q) = \underline{v}$ for $q \in (\hat{q}, 1]$, so $P_0(\cdot)$ is uniquely determined. Now for $q > \hat{q}$ equation (4) implies $(1-\rho)\underline{v} + \rho P_0(t_0((1-\mu)q)) = P_0(q) = \underline{v}$ so we must have $t_0(x) > \hat{q}$ for $x > (1-\mu)\hat{q}$. We conclude that $t_0(x) = 1$ and $R_0(x) = \left(\frac{\delta\mu}{1-\delta} + (1-x)\right)\underline{v}$ for $x \in ((1-\mu)\hat{q}, 1-\mu]$. Furthermore, since $\hat{q}$ is a steady state, we also have $t_0(x) = \hat{q}$ and $R_0(x) = \left(\frac{\delta\mu\hat{q}}{1-\delta} + (\hat{q} - q)\right)\overline{v}$, for $x \in [0, (1-\mu)\hat{q}]$. Hence $\{P_0, R_0, t_0\}$ is uniquely determined. Note that this case is characterized by $\mu = \underline{\mu}$, for at the state $(1-\mu)\hat{q}$ the monopolist must be indifferent between selecting $y = \hat{q}$ and $y = 1$.

Next, suppose that $\breve{p}_1 > \underline{v}$. Then we claim that $\breve{p}_1 = \tilde{p}_1$. To prove this claim, we will show that there exists $\varepsilon > 0$ s.t. $t_0((1-\mu)q) = \hat{q}$ for all $q \in (\hat{q}, \hat{q}+\varepsilon]$. Observe first that $t_0((1-\mu)q) < q$ for all $q \in (\hat{q}, 1]$ s.t. $P_0(q) > \underline{v}$. Indeed, $t_0((1-\mu)q) \geq q$ would imply that $P_0(t_0(1-\mu)q) \leq P_0(q)$, and so (4) would yield $P_0(q) = (1-\rho)\ \underline{v} + \rho P_0(t_0(1-\mu)q) \leq (1-\rho)\ \underline{v} + \rho P_0(q)$, or $P_0(q) \leq \underline{v}$. Now pick some $q_0 \in (\hat{q}, 1]$ with $P_0(q_0) > \underline{v}$ and iteratively define $q_k = t_0((1-\mu)q_{k-1})$. If there existed no such ε, then $\{q_k\}$ would be a decreasing sequence bounded below by $\hat{q}$. Then $P_0(q_0) = (1-\rho^k)\underline{v} + \rho^k P(q_k)$ would yield $\lim_{k\to\infty} P_0(q_0) = \underline{v}$, contradicting that $P_0(q_0) > \underline{v}$.

Now let $\tilde{q}_1 = \max\{x \leq 1-\mu : t_0(x) = \hat{q}\}$; we will show that $\tilde{q}_1 = \tilde{x}_1$. There will be three cases.

First, suppose that $\tilde{q}_1 = 1-\mu$. Then $P_0(q) = \tilde{p}_1$ for all $q \in (\hat{q}, 1]$, $R_0(x) = \tilde{R}(x)$ and $t_0(x) = \hat{q}$ for all $x \in [0, 1-\mu]$, so $\{P_0, R_0, t_0\}$ is uniquely determined. In this case we have $\tilde{x}_1 = 1-\mu$ and $m = 0$.

Next, let $\tilde{q}_1 < 1-\mu$. Define $\breve{p}_2 = \sup\{P_0(q) | q > \frac{\tilde{q}_1}{1-\mu}\}$, and consider the case where $\breve{p}_2 = \underline{v}$. Then as above, we have $P_0(q) = \underline{v}$ for $q \in (\frac{\tilde{q}_1}{1-\mu}, 1]$, $R_0(x) = R_1(x)$ and $t_0(x) = 1$ for $x \in (\tilde{q}_1, 1-\mu]$. Since $R_0(x) = \tilde{R}(x)$ for all $x < \tilde{q}_1$, the continuity of R_0 at $x = \tilde{q}_1$ then implies that $\tilde{R}(\tilde{q}_1) = R_1(\tilde{q}_1)$, i.e. $\tilde{q}_1 = x^* = \tilde{x}_1$.

Finally, let $\tilde{q}_1 < 1-\mu$ and $\breve{p}_2 > \underline{v}$. Then we claim that $\breve{p}_2 = \tilde{p}_2$. Indeed, as above we can show that there exists $\varepsilon > 0$ s.t. $t_0(q) = \frac{\tilde{q}_1}{1-\mu}$ for all $q \in (\tilde{q}_1, \tilde{q}_1+\varepsilon]$, establishing the claim. Next, we show that $\tilde{q}_1 = \tilde{x}_1$. Define $V(x) = \pi(\hat{q};x) - \frac{\mu \tilde{p}_1 x}{(1-\mu)(1-\delta)}$. Then since $t_0(x) = \hat{q}$ for $x \leq \tilde{q}_1$, and $t_0(q) = \tilde{y}_1$ for all $q \in (\tilde{q}_1, \tilde{q}_1+\varepsilon]$, we must have $V(\tilde{q}_1) = 0$. Furthermore, the definition of $\tilde{x}_1$ implies $V(\tilde{x}_1) = 0$. Since $V'(x) < 0$ we must have $\tilde{q}_1 = \tilde{x}_1$.

The same argument can now be applied inductively. Given $\tilde{q}_k = \max\{x : t_0(x) = \tilde{q}_{k-1}\}$, it must be that $\tilde{q}_k = \tilde{x}_k$. In the case where $\tilde{q}_k = 1-\mu$ we have $\tilde{x}_k = 1-\mu$ and $m = k-1$. When $\tilde{q}_k < 1-\mu$ and $\breve{p}_{k+1} = \underline{v}$ then $\tilde{q}_k = x^* = \tilde{x}_k$ and $m = k-1$. Finally, when $\breve{p}_{k+1} > \underline{v}$ then $m > k-1$ and $\tilde{q}_{k+1} = \max\{x : t_0(x) = \tilde{q}_k\}$.

We conclude that in each case $\{P_0, R_0, t_0\}$ coincides with a single instance of (A-5) and (A-6). Since each μ uniquely defines the sequence $\{\tilde{x}_k\}_{k=0}^{\infty}$, there is at most one triplet $\{P, R, t\}$, and hence at most one triplet $\{P_0, R_0, t_0\}$ associated with any $\mu \geq \underline{\mu}$. *Q.E.D.*

Proof of Corollary 2:

By construction we have $P(\tilde{y}_k) = \tilde{p}_k = (1-\rho^k)\underline{v} + \rho^k \overline{v}$ and $\tilde{y}_k = \frac{\tilde{p}_{k-1}\tilde{y}_{k-1}}{\mu \underline{v} + (1-\mu)\tilde{p}_{k-1}}$, so

$$\frac{P(\tilde{y}_k) - P(\tilde{y}_{k-1})}{\tilde{y}_k - \tilde{y}_{k-1}} = \frac{-\rho^{k-1}(1-\rho)(\overline{v}-\underline{v})}{\tilde{y}_{k-1}\left(\frac{\tilde{p}_{k-1}}{\mu \underline{v} + (1-\mu)\tilde{p}_{k-1}} - 1\right)}.$$

Using $p_{k-1} - \underline{v} = \rho^{k-1}(\overline{v}-\underline{v})$, this can be rewritten as

$$\frac{P(\tilde{y}_k) - P(\tilde{y}_{k-1})}{\tilde{y}_k - \tilde{y}_{k-1}} = \frac{-(1-\rho)(\tilde{p}_{k-1} - \mu(\tilde{p}_{k-1} - \underline{v}))}{\mu \tilde{y}_{k-1}}.$$

As z tends to zero, $\tilde{y}_k - \tilde{y}_{k-1}$ converges to zero. Hence, the above equation converges to $P'(y)y = -\frac{\lambda+r}{\lambda}P(y)$. Solving this differential equation gives $P(y) = k_0 y^{-\frac{\lambda+r}{\lambda}}$. Since $P(\hat{q}) = \overline{v}$, the particular solution has $k_0 = \overline{v}\hat{q}^{\frac{\lambda+r}{\lambda}}$. Next, upon dividing both sides by z the equation $\tilde{y}_k - \tilde{y}_{k-1} = \mu \tilde{y}_{k-1}\left(\frac{\tilde{p}_{k-1} - \underline{v}}{\mu \underline{v} + (1-\mu)\tilde{p}_{k-1}}\right)$ converges to $\dot{y} = \lambda y \frac{P(y) - \underline{v}}{P(y)} = \lambda y \left(1 - \frac{\underline{v}}{\overline{v}}\left(\frac{\hat{q}}{y}\right)^{\frac{\lambda+r}{\lambda}}\right)$. *Q.E.D.*

Proof of Theorem 3:

Suppose there is a stationary equilibrium with $y^* < \hat{q}$ as a steady state. Then by the proof of Lemma 2 we have $P(y) = \overline{v}$ for all $y \leq y^*$, $P(y) < \overline{v}$ for $y \in (y^*, \hat{q}]$, and $P(y) = \underline{v}$ for $y \in (\hat{q}, 1]$. This implies that over the interval $(y^*, 1)$ the stock is strictly increasing, i.e. $t((1-\mu)y) > y$ (see the argument at the end of the proof of Theorem 4). It follows that over the interval $(y^*, 1]$ the stationary triplet must coincide with the Coase Conjecture equilibrium triplet.

First, we show that the existence of a reputational equilibrium implies $\mu \in (\underline{\mu}, \bar{\mu}]$. To establish $\mu > \underline{\mu}$, observe that since $P(y) = \overline{v}$ for all $y \leq y^*$ we must have $t((1-\mu)y) = y^*$ for all $y \leq y^*$, and hence $R((1-\mu)y^*) = \frac{\mu y^* \overline{v}}{1-\delta}$. Furthermore, continuity of the function R implies that y^* solves $\varsigma(y) = 0$, where $\varsigma(y) = \frac{\mu y \overline{v}}{1-\delta} - R^C((1-\mu)y)$ and $R^C(\cdot)$ is the Coase Conjecture profit function. Now the definition of $\underline{\mu}$ implies $\varsigma(\hat{q}) < 0$ for all $\mu < \underline{\mu}$. As $\varsigma(\cdot)$ is strictly increasing, the existence of $y^* < \hat{q}$ therefore requires $\mu > \underline{\mu}$.

To establish that $\mu \leq \bar{\mu}$, we provide a proof by contradiction. Suppose that for some $\mu > \bar{\mu}$ a reputational equilibrium exists. Define $K = \min\{k : \bar{x}_j < \bar{x}_{j-1} \text{ for } j \leq k\}$. Since the Coase Conjecture equilibrium does not exist for $\mu > \bar{\mu}$, we necessarily have $\bar{x}_K > 0$. Furthermore, since $\bar{x}_{K+1} \geq \bar{x}_K$, when the initial state is $\bar{x}_K$ the monopolist prefers selling to $\bar{y}_K$ to selling to $\bar{y}_{K-1}$, i.e. $\mu p_K \bar{y}_K + \delta R^C(\bar{x}_K) \geq R^C(\bar{x}_K)$ where $R^C(\bar{x}_K)$ is given by (A-4). This implies that $\varsigma(\bar{y}_K) = \frac{\mu p_K \bar{y}_K}{1-\delta} - R^C(\bar{x}_K) > 0$; since $\varsigma(\cdot)$ is strictly increasing we would have to have $y^* < \bar{y}_K$. But this is impossible, as the construction (A-2)-(A-4) cannot be extended to initial states below $\bar{x}_K$.

Second, we show that a reputational equilibrium exists for any $\mu \in (\underline{\mu}, \bar{\mu}]$. For such μ we have $\varsigma(\hat{q}) > 0$. Because $\varsigma(0) = -R(0) < 0$, and $\varsigma(\cdot)$ is strictly increasing, there exists a unique $y^* \in (0, \hat{q})$ such that $\varsigma(y^*) = 0$. Now let $P(y) = \overline{v}$ for $y \in [0, y^*]$, and $R(x) = (y^* - x)\overline{v} + \delta \frac{\mu y^* \overline{v}}{1-\delta}$ and $t(x) = y^*$ for $x \in [0, (1-\mu)y^*]$, and let $\{P, R, t\}$ be given by (A-2)-(A-4) elsewhere. It is then easily checked that $\{P, R, t\}$ is a stationary triplet.

Next, we prove that $\bar{\mu} > \underline{\mu}$ for all $\delta > 0$. We will establish that if $\mu \leq \underline{\mu}$ then the sequence $\{\bar{x}_k\}$ is strictly decreasing and m is finite. The continuity of $\bar{x}_k$ in μ then implies that the same property holds in a right neighborhood of $\underline{\mu}$, so $\bar{\mu} > \underline{\mu}$.

First, we show that $\bar{x}_2 < \bar{x}_1$ holds for any $\mu \leq \underline{\mu}$. Indeed, since $\mu \leq \underline{\mu}$ iff $\frac{\mu \hat{q} \overline{v}}{1-\delta} \leq \pi(\bar{y}_0; \bar{x}_1)$, we have $h_2(\bar{x}_1) = \pi(\bar{y}_1; \bar{x}_1) - \pi(\bar{y}_0; \bar{x}_1) = P(\hat{q})\mu\hat{q} + \delta\pi(1; \bar{x}_1) - \pi(1; \bar{x}_1) \leq P(\hat{q})\mu\hat{q} - (1-\delta)\frac{\mu \hat{q} \overline{v}}{1-\delta} = -(\overline{v} - P(\hat{q}))\mu\hat{q} < 0$. Since $h_2(\bar{x}_2) = 0$ and $\frac{dh_2}{dx} < 0$ we must have $\bar{x}_2 < \bar{x}_1$.

Next, we show that there exists $\Delta > 0$ such that $\bar{x}_3 < \bar{x}_2 - \Delta$. Indeed, $h_3(\bar{x}_2) = \pi(\bar{y}_2; \bar{x}_2) - \pi(\bar{y}_1; \bar{x}_2) = P(\bar{y}_2)\frac{\mu \bar{x}_2}{(1-\mu)} + \delta\pi(\bar{y}_1; \bar{x}_2) - \pi(\bar{y}_1; \bar{x}_2) = P(\bar{y}_2)\frac{\mu \bar{x}_2}{(1-\mu)} - (1-\delta)\pi(\bar{y}_0; \bar{x}_1) - (1-\delta)(\bar{x}_1 - \bar{x}_2)\underline{v} \leq P(\bar{y}_2)\frac{\mu \bar{x}_2}{(1-\mu)} - (1-\delta)\frac{\mu \hat{q} \overline{v}}{1-\delta} - (1-\delta)(\bar{x}_1 - \bar{x}_2)\underline{v} < -(\overline{v} - P(\bar{y}_2))\frac{\mu \bar{x}_2}{(1-\mu)} - (1-\delta)(\bar{x}_1 - \bar{x}_2)\underline{v} < 0$. The first inequality follows because $\mu \leq \underline{\mu}$, the second one because $\bar{x}_2 < \bar{x}_1 = (1-\mu)\hat{q}$. Since $h_3'(x) = -P(\bar{y}_2) + P(\bar{y}_1)$ is independent of x, it follows that $\bar{x}_2 - \bar{x}_3 = \frac{h_3(\bar{x}_2)}{h_3'} > \frac{(1-\delta)(\bar{x}_1 - \bar{x}_2)\underline{v}}{P(\bar{y}_2) - P(\bar{y}_1)} = \Delta > 0$.

Finally, we establish that for each $k \geq 3$ we have $\bar{x}_{k-1} - \bar{x}_k \geq \Delta$, implying $m \leq 2 + \frac{1-\bar{x}_2}{\varepsilon}$. We have shown above that the result is true at $k = 3$. Next, we show that if the result for some $k \geq 3$ then it also holds for $k+1$. Indeed, $\pi(\bar{y}_{k-1}; \bar{x}_k) \geq \pi(\bar{y}_0; \bar{x}_k) = (1 - \bar{x}_k)\underline{v} + \delta R(1-\mu) = \pi(\bar{y}_0; \bar{x}_1) + (\bar{x}_1 - \bar{x}_k)\underline{v}$. Therefore, following the argument in the previous paragraph, we have

$h_{k+1}(\bar{x}_k) < -(1-\delta)(\bar{x}_1 - \bar{x}_k)\underline{v} < 0$. Since $h'_{k+1}(x) = -P(\bar{y}_k) + P(\bar{y}_{k-1})$ is independent of x, and since $h_{k+1}(\bar{x}_{k+1}) = 0$ we obtain $\bar{x}_k - \bar{x}_{k+1} = \frac{h_{k+1}(\bar{x}_k)}{h'_{k+1}} > \frac{(1-\delta)(\bar{x}_1 - \bar{x}_k)\underline{v}}{P(\bar{y}_{k-1}) - P(\bar{y}_k)} > \Delta$. *Q.E.D.*

Proof of Lemma 3:

Define $\xi_k = \frac{\bar{x}_k}{\bar{x}_{k-1}}$. It then follows from (A-1) that

$$\xi_k = (1-\mu)^{-1}\Big(1 - (\xi_{k-1}^{-1} - 1)\delta^{-(k-2)}\psi_k\Big), \tag{A-7}$$

where $\psi_k = \frac{\bar{v}}{(1-\mu)^{k-2}(\bar{v} - \underline{v})} > 1$. Fix $\mu \leq \bar{\mu}(\delta)$ so that $\xi_k(\delta) < 1$ for all $k = 1, ..., m+1$. Next, observe that ξ_2 is independent of δ. Thus, (A-7) implies that ξ_3 is increasing in δ, i.e. $\xi_3(\delta) < 1$ implies $\xi_3(\delta') < \xi_3(\delta) < 1$ for all $\delta' < \delta$. Next, suppose that $\xi_{k-1}(\delta') < \xi_{k-1}(\delta) < 1$; it then follows from (A-7) that $\xi_k(\delta') < \xi_k(\delta) < 1$. We conclude that if $\mu \leq \bar{\mu}(\delta)$ then $\mu < \bar{\mu}(\delta')$ for all $\delta' < \delta$, so $\bar{\mu}$ is a decreasing function of δ.

Next, we argue that $\bar{\mu}(1) > 0$. Indeed, when $\delta = 1$ and $\mu = 0$ the difference equation (A-1) reduces to

$$\bar{x}_k - \bar{x}_{k-1} = (\bar{x}_{k-1} - \bar{x}_{k-2})\psi,$$

where $\psi = \frac{\bar{v}}{\bar{v} - \underline{v}} > 1$. Hence $\bar{x}_k - \bar{x}_{k-1} = \psi^{k-1}(\bar{x}_2 - \bar{x}_1) < 0$. Let m be such that $\bar{x}_m < 0 \leq \bar{x}_{m-1}$. Then since $\bar{x}_k$ is a continuous function of μ, it follows that there exists $\mu' > 0$ such that for all $\mu \in [0, \mu')$ we have $\bar{x}_k - \bar{x}_{k-1} > 0$ and $\bar{x}_m < 0 \leq \bar{x}_{m-1}$. *Q.E.D.*

Proof of Theorem 4:

For all $i = 1, ..., N-1$, analogously to the two-step case, given q_i as a steady state construct a sequence $\{\tilde{x}_k^i\}_{k=0}^{\infty}$ to the right of q_i as follows. Let $\tilde{x}_0^i = (1-\mu)\, q_i$ and let $\tilde{P}^i(q) = (1-\rho) f(q) + \rho f(q_i)$ for $q \geq q_i$. Given $\tilde{P}^i(\cdot)$, let $\tilde{x}_1^i$ be the highest state such that the seller prefers selecting $y = q_i$ to selecting $y \in (q_i, \tilde{y}_1^i]$. For $k \geq 2$ let $\tilde{t}^i(q)$ be the seller's optimal choice over $(\tilde{y}_{k-2}^i, \tilde{y}_{k-1}^i]$ and redefine $\tilde{P}^i(q) = (1-\rho) f(q) + \rho f\left(\tilde{t}^i(q)\right)$ for $q \geq \tilde{y}_{k-1}^i$. Finally, let $\tilde{x}_k^i$ be the highest state such that the seller prefers selecting $y \in (\tilde{y}_{k-2}^i, \tilde{y}_{k-1}^i]$ to selecting $y \in (\tilde{y}_{k-1}^i, \tilde{y}_k^i]$. Let $\tilde{R}^i(q)$ be the seller's payoff function associated with $\tilde{P}^i(q)$.

Also analogously to the two-step case, given q_i as a steady state construct a sequence $\{\bar{x}_k^i\}$ to the left of q_i as follows. Let $\bar{x}_0^i = (1-\mu)\, q_i$ and let $\bar{P}^i(q) = (1-\rho) f(q) + \rho f(q_i)$ for $q \leq q_i$. Given $\bar{P}^i(\cdot)$, let $\bar{x}_1^i$ be the smallest state such that the seller prefers selecting $y = q_i$ to selecting $y \in (\bar{y}_1^i, q_{i-1}]$. For $k \geq 2$ let $\bar{t}^i(q)$ be the seller's optimal choice over $(\bar{y}_{k-1}^i, \bar{y}_{k-2}^i]$ and redefine $\bar{P}^i(q) = (1-\rho) f(q) + \rho f\left(\bar{t}^i(q)\right)$ for $q < \bar{y}_{k-1}^i$. Finally, let $\bar{x}_k^i$ be the smallest state such that the seller prefers selecting $y \in (\bar{y}_{k-1}^i, \bar{y}_{k-2}^i]$ to selecting $y \in (\bar{y}_k^i, \bar{y}_{k-1}^i]$. Let m^i be the largest value of k

such that $0 \leq \bar{x}_j^i < \bar{x}_{j-1}^i$ for all $j \leq k$, and let $\bar{R}^i(q)$ be the seller's payoff function associated with $\bar{P}^i(q)$.

Let $A_0 = \{i | \lim_{k\to\infty} \tilde{x}_k^i \geq 1-\mu\}$. If $A_0 = \phi$, let $q_{i_0} = 1$. Otherwise, let $i_0 = \min A_0$. Then define $P(q) = \tilde{P}^{i_0}(q)$ for $q > q_{i_0}$. Also let $R(q)$ and $t(q)$ be the equilibrium value function and policy function associated with $P(q)$.

(P) Now we will construct $P(q)$ for $q \leq q_{i_0}$. Define i_1 such that $\lim_{k\to\infty} \tilde{x}_k^{i_1} \geq \lim_{k\to\infty} \tilde{x}_k^i$ for all $i < i_0$. Note that $\lim_{k\to\infty} \tilde{x}_k^{i_1} > (1-\mu) q_{i_0-1}$ because $\lim_{k\to\infty} \tilde{x}_k^{i_0-1} > (1-\mu) q_{i_0-1}$. If $\bar{R}^{i_0}((1-\mu) q_{i_0-1}) < \tilde{R}^{i_1}((1-\mu) q_{i_0-1})$, let q' be such that $\bar{R}^{i_0}((1-\mu) q') = \tilde{R}^{i_1}((1-\mu) q')$, and set $P(q) = \bar{P}^{i_0}(q)$ for $q \in (q', q_{i_0}]$ and $P(q) = \tilde{P}^{i_1}(q)$ for $q \in (q_{i_1}, q']$. Now return to (P) with i_1 taking the role of i_0. (P') If $\bar{R}^{i_0}((1-\mu) q_{i_0-1}) \geq \tilde{R}^{i_1}((1-\mu) q_{i_0-1})$ a proof similar to the one showing $\underline{\mu} < \bar{\mu}$ in the two-step case yields $\bar{x}_{m^{i_0}} \leq q_{i_0-2}$. Hence, $\bar{P}^{i_0}(q)$ exists for $q \in (q_{i_0-2}, q_{i_0-1}]$. Define i_2 such that $\lim_{k\to\infty} \tilde{x}_k^{i_2} \geq \lim_{k\to\infty} \tilde{x}_k^i$ for all $i < i_0 - 1$. If $\bar{R}^{i_0-1}((1-\mu) q_{i_0-2}) < \tilde{R}^{i_2}((1-\mu) q_{i_0-2})$, let q' be such that $\bar{R}^{i_0-1}((1-\mu) q') = \tilde{R}^{i_2}((1-\mu) q')$, and set $P(q) = \bar{P}^{i_0-1}(q)$ for $q \in (q', q_{i_0-1}]$ and $P(q) = \tilde{P}^{i_2}(q)$ for $q \in (q_{i_2}, q']$. Now return to (P) with i_2 taking the role of i_0. If $\bar{R}^{i_0-1}((1-\mu) q_{i_0-2}) \geq \tilde{R}^{i_2}((1-\mu) q_{i_0-2})$ then return to (P') with i_2 taking the role of i_1 and q_{i_0-2} taking the role of q_{i_0-1}. *Q.E.D.*

Proof of Theorem 5:

(i) Let $\pi^* = \max_{q\in[0,1]} qf(q)$ be the monopoly profit, and suppose that $\pi^* > q_N f(q_N) = v_N$. Define $\underline{\mu}$ as the solution to the equation $f(\mu) = 0$, where $f(\mu) = \frac{\mu\pi^*}{1-\delta} - (1-(1-\mu)q_{N-1})v_N - \frac{\delta\mu v_N}{1-\delta}$. Then since $f(1) = \frac{\pi^* - v_N}{1-\delta}^* > 0$ and since $f(0) = -(1-q_{N-1})v_N < 0$, we have $0 < \underline{\mu} < 1$. Furthermore, note that $y \in [0, q_{N-1}]$ and $\mu < \underline{\mu}$ imply $\frac{\mu y f(y)}{1-\delta} < (1-(1-\mu)y)v_N + \frac{\delta\mu v_N}{1-\delta}$. Suppose that $\mu < \underline{\mu}$ and that there existed a stationary equilibrium with steady state $y_s \in [0, q_{N-1}]$. Let $\hat{y}$ be be the largest such steady state. Then when the state is $q = (1-\mu)\hat{y}$ the monopolist would earn $\frac{\mu\hat{y}f(\hat{y})}{1-\delta}$. If instead the monopolist were to select $y = 1$ and remain there forever after, she would earn $(1-(1-\mu)\hat{y})v_N + \frac{\delta\mu v_N}{1-\delta}$, which exceeds $\frac{\mu\hat{y}f(\hat{y})}{1-\delta}$ for $\mu < \underline{\mu}$. We conclude that $y = 1$ is the unique steady state when $\mu < \underline{\mu}$. Thus the Coase Conjecture equilibrium is the unique equilibrium.

(ii) Given a stationary triplet, we say that a steady state y_s is reachable from $(1-\mu) q^*$ if $\lim_{n\to\infty} g^n((1-\mu) q^*) = y_s$, where $g(x) = (1-\mu) t(x)$. Note that a reachable steady state from $(1-\mu) q^*$ is the smallest steady state above q^*. Define $q_1^* = \min \arg\max_{q_i > q^*} q_i f(q_i)$.

First we claim that there exists $\bar{\mu}_1 < 1$, such that q_1^* cannot be a reachable steady state from $(1-\mu) q^*$ for any $\mu > \bar{\mu}$. Suppose not; let $P(q)$ be the acceptance function associated with some stationary equilibrium having q_1^* as a reachable steady state. Then for $x \in [(1-\mu) q^*, (1-\mu) q_1^*]$, it must be the case that $t(x) \notin (q^*, q_1^*)$. Indeed, let $q^* = q_{i-K} < ... < q_i = q_1^*$, and suppose that $K > 1$. Analogously to the two-step case construct a sequence $\{\bar{x}_k\}_{k=0}^{\infty}$ starting at the steady state q_1^*, i.e. $\bar{x}_0 = (1-\mu) q_1^*$. First we show that $t(x) \neq q_{i-1}$. If $t(x) = q_{i-1}$ then by the definition of $q_1^* = q_i$ we would have $\bar{x}_1 = (1-\mu) q_{i-1}$ and $\bar{x}_2 = \frac{q_{i-1}v_{i-1} - q_i v_i}{v_{i-1} - v_i} \leq 0$, yielding a contradiction $t(x) = q_i$. Next, for $K > 2$ we show that $t(x) \neq q_{i-2}$. Since q_1^* is a reachable steady state, we must have $t((1-\mu) q) > q$ for $q < q_1^*$. Thus $t((1-\mu) q_{i-2}) > q_{i-2}$. Since $t((1-\mu) q_{i-2}) \neq q_{i-1}$ we have $t((1-\mu) q_{i-2}) = q_i$ and $P((1-\mu) q_{i-2}) = (1-\rho) v_{i-2} + \rho v_i$. If $t(x) = q_{i-2}$ then we would have $\bar{x}_1 = (1-\mu) q_{i-2}$ and $\bar{x}_2 = \frac{q_{i-2}v_{i-2} - q_i v_i}{v_{i-2} - v_i} \leq 0$, a contradiction. Using a similar argument we conclude that $t(x) \notin \{q_{i-K+1}, ..., q_{i-1}\}$. Hence, $P((1-\mu) q^*) = (1-\rho) v_{i-K} + \rho v_i$. Define $\bar{\mu}_1$ as the solution to $\bar{x}_2 = \frac{q_{i-K}v_{i-K} - q_i v_i}{v_{i-K} - v_i} = (1-\bar{\mu}_1) q_{i-K}$. Note that $\bar{\mu}_1 = \frac{(q_1^* - q^*) f(q_1^*)}{q^* \left(f(q^*) - f\left(q_1^*\right)\right)} < 1$. Then for $\mu > \bar{\mu}_1$, $(1-\mu) q_{i-K} < \bar{x}_2$ and $t((1-\mu) q_{i-K}) \leq t(\bar{x}_2) \leq q_{i-K}$, contradicting that q_1^* is a reachable steady state from $(1-\mu) q^*$.

Secondly given $q_1^*, ..., q_{k-1}^*$ define $q_k^* = \min \arg\max \{q_i f(q_i) : q_i > q^* \text{ and } q_i \notin \{q^*, q_1^*, ..., q_{k-1}^*\}\}$ and let $q' = \max\{q_i < q_k^* : q_i f(q_i) > q_k^* f(q_k^*)\}$. Mimicking the argument above, we can show for any $x \in [(1-\mu) q', (1-\mu) q_k^*]$, it must be the case that $t(x) \notin (q', q_k^*)$. Letting $\bar{\mu}_k = \frac{(q' - q_k^*) f(q')}{q' \left(f(q') - f\left(q_k^*\right)\right)} < 1$, it follows that for any $\mu > \bar{\mu}_k$, q_k^* is not a reachable steady state from $(1-\mu) q^*$.

Since the inductive step stops after finitely many iterations, we conclude that no $q_i > q^*$ can be a reachable steady state from $(1-\mu) q^*$ when $\mu > \bar{\mu} = \max\{\bar{\mu}_k\}$. Therefore $t((1-\mu) q^*) \leq q^*$, implying $P(q^*) \geq f(q^*)$. We conclude that when $\mu > \bar{\mu}$, we have $R(0) \geq q^* f(q^*) + \frac{\delta \mu q^* f(q^*)}{1-\delta}$. Since the outcome path of any stationary equilibrium can always be duplicated in a rental equilibrium by selecting an appropriating sequence of rentals, and since the rental profit is uniquely maximized by charging $f(q^*)$, we also must have $R(0) \leq q^* f(q^*) + \frac{\delta \mu q^* f(q^*)}{1-\delta}$. We conclude that $R(0) = q^* f(q^*) + \frac{\delta \mu q^* f(q^*)}{1-\delta}$ and that $t(q) = q^*$ for all $q \leq (1-\mu) q^*$. Thus in any stationary equilibrium q^* is the unique steady state reachable from 0.

(iii) We start by constructing $\mu_L \in (\underline{\mu}, \bar{\mu})$ such that at μ_L there exists a reputational equilibrium that results in the monopoly outcome (from $q = 0$). For this purpose, define the fictitious demand curve $\widetilde{f}(q) = \min\{f(q), f(q_{k+1})\}$, where $q_k = q^*$ for some $k < N$. By Theorem 4 there exists a stationary equilibrium for the demand curve $\widetilde{f}$. Let $g(\mu)$ denote the profit in this equilibrium when the initial state equals $(1-\mu)q^*$. Note that since the equilibrium constructed in Theorem 4 is the most profitable one, $g(\mu)$ can only jump up as μ increases, and hence is upper semicontinuous and continuous from the right. Additionally, let $h(\mu) = \frac{\mu q^* f(q^*)}{1-\delta}$ be the profit from serving replacement demand when monopoly is a steady state. Note that $h(\cdot)$ is continuous in μ, and that $h(0) < g(0)$ and $h(1) > g(1)$, so $\text{Im}(g) \subset \text{Im}(h)$. Define $r(\mu) = h^{-1}(g(\mu))$. Then $r : [0,1] \to [0,1]$, and so by the Lemma in Roberts and Sonnenschein (1976) has a fixed point μ_L. Furthermore, it follows from right continuity that $g(\mu) < h(\mu)$ in a right neighborhood of μ_L.

We now claim that when the demand curve is $f(\cdot)$ and $\mu = \mu_L$ there exists a stationary equilibrium in which the monopolist earns $\frac{\mu_L q^* f(q^*)}{1-\delta}$ when the state is $(1-\mu)q^*$. Indeed, let $P(q) = (1-\rho)f(q) + \rho f(q^*)$ for $q \leq q^*$, and for $q > q^*$ let $P(q)$ be given by the one computed in Theorem 4 for the demand curve $\widetilde{f}(\cdot)$. Then at $q = (1-\mu)q^*$ the monopolist is indifferent between staying at q^* forever and moving beyond q^*, thereby earning the continuation profit $g(\mu_L) = h(\mu_L) = \frac{\mu_L q^* f(q^*)}{1-\delta}$. Meanwhile, for $q > (1-\mu)q^*$ the profits from going to q^* and remaining there ever after fall below the profits from going forward (see the argument following (A-4)) in the proof of Theorem 1, so the equilibrium is a reputational one.

Next, we show that there exists $\mu_H > \mu_L$ such that for every $\mu \in (\mu_L, \mu_H)$ there exists a reputational equilibrium whose smallest steady state satisfies $y^* < q^*$. Let $\{P_0(\cdot), R_0(\cdot), t_0(\cdot)\}$ denote the stationary equilibrium constructed in the proof of Theorem 4 when the demand function is $\widetilde{f}(\cdot)$. For all $q \in [0, 1-\mu]$ define $R_1(q) = \max_{y > q^*}\{P_0(y)(y-q) + \delta R_0((1-\mu)y))$, the maximal profit attainable when the demand curve is $f(\cdot)$, the state is q, the monopolist is constrained to select a state $y > q^*$, and the equilibrium $\{P_0(\cdot), R_0(\cdot), t_0(\cdot)\}$ is played thereafter. Let $t_1(q)$ denote the minimum element of the corresponding argmax, and let $P_1(q) = (1-\rho)f(q) + \rho P_0(t_1((1-\mu)q)$. Let $R_2(q) = \max_{y \leq q^*}\{P_1(y)(y-q) + \delta R_1((1-\mu)y))$ the maximal profit when the acceptance function is P_1, and monopolist is constrained to select $y \in [0, q^*]$, and let $t_2(q)$ be the corresponding policy function. Finally, let $\widetilde{y} = \max\{q \in [0, q^*] : R_2(q) \geq R_1(q)\}$. Then provided $\widetilde{y} < q^*$, the triple $\{P_1(\cdot), R_1(\cdot), t_1(\cdot)\}$ defines a stationary equilibrium for $q > \widetilde{y}$ (see Deneckere and Liang, 2006).

We now claim that there exists $\mu_H^1 > \mu_L$ and $\bar{y} \in (q_{k-1}, q^*)$ such that $\widetilde{y}(\mu) \leq \bar{y}$ for all $\mu \in (\mu_L, \mu_H^1)$. Let $p^* = (1-\rho)f(q^*) + \rho f(q_{k+1})$; note that p^* is an upper bound to $P_1(q)$ for all $q > q_{k-1}$. Consequently, $H(q) = (q^* - (1-\mu)q)p^* + \delta R_1((1-\mu)q^*) \geq R_2(q)$. Observe now that at $\mu = \mu_L$ we have $R_1((1-\mu_L)q^*) = R_0((1-\mu_L)q^*) = g(\mu_L) = h(\mu_L) = \frac{\mu_L q^* f(q^*)}{1-\delta} = H(q^*) + \mu_L q^*(f(q^*) - p^*) > H(q^*)$. Since R_1 and H are continuous in μ, there exists $\mu_H^1 > \mu_L$ and $\bar{y} \in (q_{k-1}, q^*)$ such that $R_1(\bar{y}) \geq H(\bar{y}) \geq R_2(\bar{y})$, and hence $\widetilde{y} \leq \bar{y}$ for all $\mu \in (\mu_L, \mu_H^1)$.

Finally, we establish that there exists $\mu_H \in (\mu_L, \mu_H^1)$ such that $\frac{\mu \bar{y} f(\bar{y})}{1-\delta} < R_1((1-\mu)\bar{y})$ for all $\mu \in (\mu_L, \mu_H)$. Indeed, since $R_1((1-\mu_L)q^*) = \frac{\mu_L q^* f(q^*)}{1-\delta}$, and since $R_1(q)$ and $\frac{\mu q f(q^*)}{1-\delta}$ are decreasing and increasing functions of q, respectively, we have $\frac{\mu_L \bar{y} f(\bar{y})}{1-\delta} < R_1((1-\mu_L)\bar{y})$. The existence of μ_H

then follows from the continuity of each of these functions in μ.

We conclude that for every $\mu \in (\mu_L, \mu_H)$ there exists a stationary equilibrium $\{P_1(\cdot), R_1(\cdot), t_1(\cdot)\}$ for states above $\bar{y}$ such that $R_1((1-\mu)\bar{y}) > \frac{\mu \bar{y} f(\bar{y})}{1-\delta}$ and $R_1((1-\mu)q^*) < \frac{\mu q f(q^*)}{1-\delta}$, so there exists $y^* \in (\bar{y}, q^*)$ for which $R_1((1-\mu)y^*) = \frac{\mu y^* f(y^*)}{1-\delta}$. The acceptance function $P(q) = (1-\rho)f(q) + \rho f(y^*)$ for $q \leq y^*$ and $P(q) = P_1(q)$ for $q > y^*$ then defines a reputational equilibrium in which $t(q) = y^*$ for all $q \leq (1-\mu)y^*$ and $t(q) = t_1(q)$ for $q > (1-\mu)y^*$.

(iv) Consider any stationary equilibrium with profit function R. Since q^* cannot be a steady state, we must have $\frac{\mu q^* f(q^*)}{1-\delta} < R((1-\mu)q^*)$. For $q < q^*$, we have $\frac{\mu q f(q)}{1-\delta} < \frac{\mu q^* f(q^*)}{1-\delta} < R((1-\mu)q^*) < R((1-\mu)q)$. Therefore no $q < q^*$ can be a steady state. *Q.E.D.*