Classnote 02 (update)

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The Nash-Equilibrium Correspondence Has a Closed Graph

$$G_1(\lambda): \begin{array}{cc} L & R \\ \hline \multicolumn{1}{|c|}{1+\lambda} & \multicolumn{1}{c|}{1-\lambda} \\ \hline \end{array}$$

Let x denote the probability that the decision maker plays L, and consider the optimal x for each λ. This defines the Nash-equilibrium correspondence for this one-player game.

$$x^*(\lambda) \in E(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda < 0 \\ [0,1] & \text{if} \quad \lambda = 0 \\ 1 & \text{if} \quad \lambda > 0 \end{cases}.$$

Two Important Properties:

Proposition 1 *The graph of the Nash correspondence has a closed graph (is upper hemi-continous), i.e., for any sequence (λ^n, x^n) belonging to the graph of the correspondence and converging to some (λ, x), the limit (λ, x) belongs to the graph of correspondence.*

Proposition 2 *The correspondence may not be "lower hemi-continuous". That is, there may exist (λ, x) belonging to the graph of the correspondence and a sequence $\lambda^n \to \lambda$ such that there exists no x^n such that (λ^n, x^n) belongs to the graph of the correspondence and $x^n \to x$. Here, take $\lambda = 0$ and $x \in (0,1)$.*

$$G_2(\lambda): \begin{array}{c|c|c|} & L & R \\ \hline U & 1,1 & 0,0 \\ \hline D & 0,0 & \lambda, 2 \\ \hline \end{array}, \qquad G_3(\lambda): \begin{array}{c|c|c|} & L & R \\ \hline U & 1,1 & 0,0 \\ \hline D & 0,0 & \lambda, \lambda \\ \hline \end{array}$$

Let p denote the probability of U and q denote the probability of L, the Nash equilibrium correspondence for $G_2(\lambda)$:

$$(p^*(\lambda), q^*(\lambda)) \in E(\lambda) = \begin{cases} \{(1,1)\} & \text{if} \quad \lambda < 0 \\ \{(1,1), \left(x, \frac{\lambda}{1+\lambda}\right), (0,0) : x \in [0, \frac{2}{3}]\} & \text{if} \quad \lambda = 0 \\ \{(1,1), \left(\frac{2}{3}, \frac{\lambda}{1+\lambda}\right), (0,0) & \text{if} \quad \lambda > 0 \end{cases}$$

the Nash equilibrium correspondence for $G_3(\lambda)$:

$$(p^*(\lambda), q^*(\lambda)) \in E(\lambda) = \begin{cases} \{(1,1)\} & \text{if} \quad \lambda < 0 \\ \{(1,1), \left(\frac{\lambda}{1+\lambda}, \frac{\lambda}{1+\lambda}\right), (0,0) & \text{if} \quad \lambda \geq 0 \end{cases}$$

One additional property.

Proposition 3 *(Wilson's (1971) Oddness Theorem) Almost all finite games have a finite and odd number of equilibria.*

Properties on Sequential Equilibrium

Definition 4 *An assessment is a pair* (b, μ) *where* b *is a behavior strategy combination and* μ *is a system of beliefs.*

Definition 5 *An assessment* (b, μ) *is consistent if there exists a sequence* $\{b^{\varepsilon}, \mu^{\varepsilon}\}_{\varepsilon \downarrow 0}$ *where* b^{ε} *is a completely mixed behavior strategy combination and* μ^{ε} *is the system of beliefs generated by* b^{ε} *such that*

$$\lim_{\varepsilon \downarrow 0} (b^{\varepsilon}, \mu^{\varepsilon}) = (b, \mu)$$

Definition 6 *A sequential equilibrium is a consistent assessment* (b, μ) *for which* b *is a sequential best reply against* (b, μ).

Proposition 7 *Every sequential equilibrium is subgame perfect.*

Definition 8 *Let* Γ *be an extensive form game. If* η *is a mapping which assigns to every choice in* Γ *a positive number* η_c *such that* $\sum_{c \in C_u} \eta_c < 1$ *for every information set* u*, then the perturbed game* (Γ, η) *is the extensive form game with the same structure as* Γ*, but in which every player* i *is only allowed to use behavior strategies* b_i *which satisfy* $b_{iu}(c) \geq \eta_c$ *for all* $u \in U_i$ *and* $c \in C_u$.

Definition 9 b *is a perfect equilibrium of* Γ *if* b *is a limit point of a sequence* $\{b(\eta)\}_{\eta \downarrow 0}$ *where* $b(\eta)$ *is an equilibrium of* (Γ, η).

Example 1: A Perfect equilibrium of Γ need not be a perfect equilibrium of $N(\Gamma)$ (the normal form of Γ).

Example 2: A Perfect equilibrium of $N(\Gamma)$ is not necessarily a perfect euqilibrium of $Gamma$.

Example 3: An unreasonable subgame perfect equilibrium could be eliminated by perfect requirements

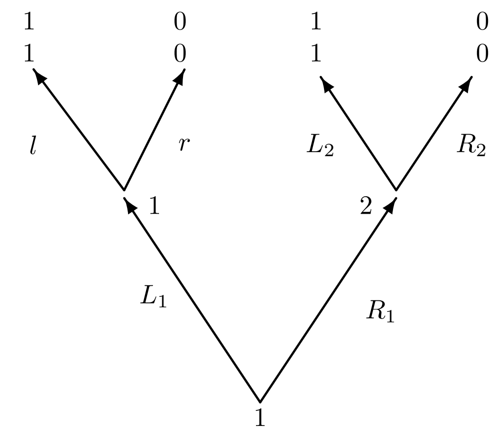
Example 3-1: An unreasonable subgame perfect equilibrium could pass perfect and sequential requirements.

Example 4: A sequential equilibrium need not be a perfect equilibrium

Proposition 10 *(Kreps and Wilson [1982])*

1. *Every perfect equilibrium is sequential.*
2. *For almost all extensive form games, almost all sequential equilibria are perfect.*
3. *For almost all games, the set of sequential equilibrium outcomes (i.e. the set of probability distributions over the endpoints resulting from sequential equilibria) coincides with the set of perfect equilibrium outcomes.*

Example 1

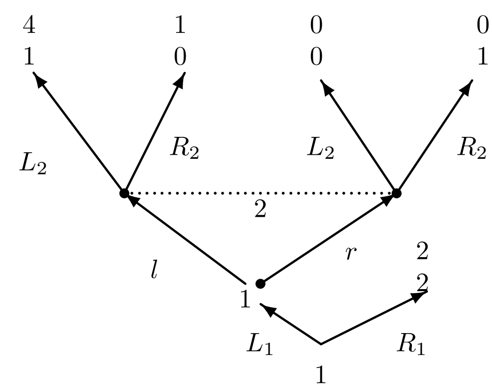


Proposition 11 *(Van Damme (1984), Kohlberg and Mertens (1986))*

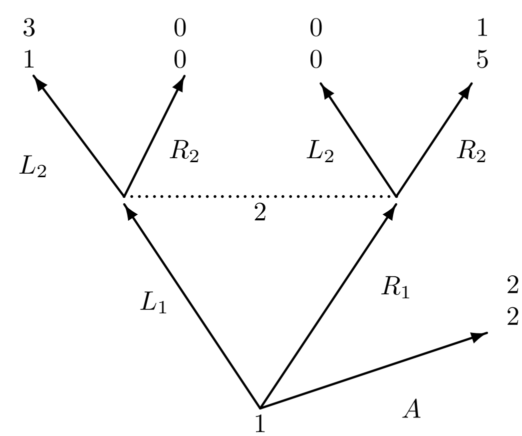
1. *If s is a proper equilibrium of $N(\Gamma)$, then $\boldsymbol{P}^s$ is a sequential equilibrium outcome in Γ.*

2. *If b is a limit point of a sequence $\{b^\varepsilon\}_{\varepsilon\downarrow 0}$ where b^ε is induced by an ε-proper equilibrium s^ε of $N(\Gamma)$, then b is a sequential equilibrium of Γ.*

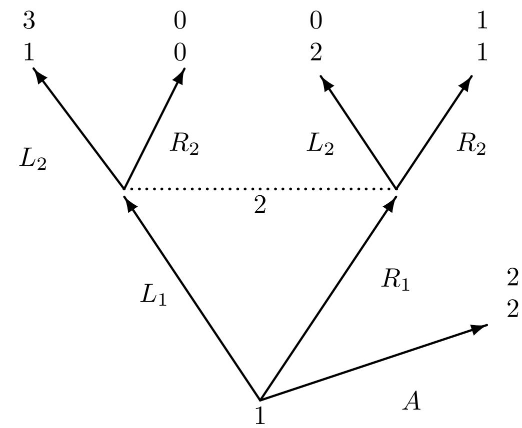
Example 2



Example 3



Example 3-1



Example 4