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March 2009

Kohlberg and Mertens (Econometrica 1986)

We will use the term "(game) tree" for the extensive form of a game with perfect recall (i.e., where every player remembers whatever he knew previously, including his past actions).

The agent normal form (Selten) of a tree is the normal form of the game between agents, obtained by letting each information set be manned by a different agent, and by giving any agent of the same player that player's payoff. A behavioral strategy of a player in a tree is a list of (mixed) strategies, one for each of his agents. Kuhn (1953) has shown that every mixed strategy of a player in a tree is equivalent to some behavioral strategy, in the sense that both give the same probability distribution on the endpoints whatever be the strategies of all opponents.

A sequential equilibrium (Kreps-Wilson) of an n-player tree is an n-tuple of behavioral strategies which is the limit of a sequence (σ_m) of completely mixed (i.e., strictly positive) behavioral strategies, such that every agent maximizes his expected payoff given the strategies of all other agents and given the limiting conditional probability distribution on his information set implied by (σ_m).

An ε-perfect equilibrium of a normal form game (Selten) is a completely mixed strategy vector, such that any pure strategy which is not a best reply has weight less than E.

An ε-proper equilibrium of a normal form game (Myerson) is a completely mixed strategy vector, such that whenever some pure strategy s_1, is a worse reply than some other pure strategy s_2,, the weight on s_1, is smaller than ε times the weight on s_2,.

A perfect (proper) equilibrium of a normal form game is a limit ($\varepsilon \to 0$) of ε-perfect (proper) eqilibria.

A perfect (proper) equilibrium of a tree is a perfect (proper) equilibrium of its agent normal form.

It is evident that "proper" is a stronger requirement than "perfect." It is also easy to verify that a perfect equilibrium of a tree is sequential (Kreps-Wilson).

Existence theorems have been proved for all the above concepts (Kreps-Wilson, Myerson, Selten).

Proposition 1 *For generic extensive-form games, the set of Nash equilibrium outcomes is finite and all Nash equilibrium within a given connected component induce the same outcome.*

Proposition 2 *Every game has at least one stable component.*

1 Beer Quiche

Consider the following version of the Beer Quiche Game. First Nature selects whether player 1, the sender, will be of the strong (s) or weak (w) type. The probability with which the strong type is selected equals $\frac{9}{10}$, and this value is common knowledge between the two players. Player 1 then sends a signal, whether to have Beer of Quiche for breakfast. Upon seeing this signal, Player 2, the receiver, then decides whether or not to challenge player 1 to a duel.

Player 1 knows that the strong type prefers beer, and that the weak type prefers quiche. More precisely, each type gets a utility of +1 from having its most preferred breakfast. Player 1 also know that each type does not like to duel. Not having to duel yields each sender type an additional utility of +2.

Player 2 profits from challenging if and only if he faces the weak type. Thus Player 2 gets a utility of +1 if he faces the weak type and duels, and a utility of −1 if he faces the strong type and duels.

The payoffs to this signalling game are summarized in the table below:

	d	n
s	$1, -1$	$3, 0$
w	$0, 1$	$2, 0$
	B	

	d	n
s	$0, -1$	$2, 0$
w	$1, 1$	$3, 0$
	Q	

The attached picture draws the extensive form of this signalling game.

We can also compute the normal form of this two-player game, as follows. A strategy for the sender is a pair of signals from the set $\{B, Q\}$, the first one describing the choice of the strong sender type, and the second one describing the choice of the weak sender type. A strategy for the receiver is a pair of actions from the set $\{d, n\}$, the first one describing his choice when he sees the sender selected beer, and the second one describing his choice when he sees the sender selected quiche. Note that we are treating the sender as a single player, so the normal form computes the expected payoff of player 1 and 2, conditional on a pure strategy profile. We then obtain:

	dd	dn	nd	nn
BB	$\frac{9}{10}, -\frac{4}{5}$	$\frac{9}{10}, -\frac{4}{5}$	$\frac{29}{10}, 0$	$\frac{29}{10}, 0$
BQ	$1, -\frac{4}{5}$	$\frac{6}{5}, -\frac{9}{10}$	$\frac{14}{5}, \frac{1}{10}$	$3, 0$
QB	$0, -\frac{4}{5}$	$\frac{9}{5}, \frac{1}{10}$	$\frac{1}{5}, -\frac{9}{10}$	$2, 0$
QQ	$\frac{1}{10}, -\frac{4}{5}$	$\frac{21}{10}, 0$	$\frac{1}{10}, -\frac{4}{5}$	$\frac{21}{10}, 0$

2 Nash Equilibria of the Normal Form

We start by computing the Nash equilibria for the normal form. Observe that the strategy dd for the receiver is strictly dominated by the strategy nn. Hence the above game is strategically equivalent to the game in which dd is eliminated as a receiver strategy:

	dn	nd	nn
BB	$\frac{9}{10}, -\frac{4}{5}$	$\frac{29}{10}, 0$	$\frac{29}{10}, 0$
BQ	$\frac{6}{5}, -\frac{9}{10}$	$\frac{14}{5}, \frac{1}{10}$	$3, 0$
QB	$\frac{9}{5}, \frac{1}{10}$	$\frac{1}{5}, -\frac{9}{10}$	$2, 0$
QQ	$\frac{21}{10}, 0$	$\frac{1}{10}, -\frac{4}{5}$	$\frac{21}{10}, 0$

We now claim that in the remaining game the strategy QB for the sender is strictly dominated by the strategy $\frac{1}{7}BB + \frac{6}{7}QQ$. Indeed, the latter strategy yields an expected payoff vector of $(\frac{27}{14}, \frac{1}{2}, \frac{31}{14})$, which strictly dominates the payoff vector from using QB, $(\frac{9}{5}, \frac{1}{5}, 2)$ that results from using QB. Hence we may again reduce the game by eliminating the strategy QB for the sender. This results in the 3×3 game:

	dn (p_1)	nd (p_2)	nn (p_3)
BB (q_1)	$\frac{9}{10}, -\frac{4}{5}$	$\frac{29}{10}, 0$	$\frac{29}{10}, 0$
BQ (q_2)	$\frac{6}{5}, -\frac{9}{10}$	$\frac{14}{5}, \frac{1}{10}$	$3, 0$
QQ (q_3)	$\frac{21}{10}, 0$	$\frac{1}{10}, -\frac{4}{5}$	$\frac{21}{10}, 0$

Note that in this game no remaining strategy for the sender or the receiver is strictly dominated.[1] However, the strategy dn is weakly dominated for the receiver (by the strategy nn). Let us now find all the Nash equilibria of this 3×3 game.

Let p_1 denote the probability with which the receiver selects dn, p_2 the probability with which the receiver selects nd, and $p_3 = 1 - p_1 - p_2$ the probability with which the receiver selects nn. Then the expected payoff of the sender equals

Pure Strategy	Expected Payoff to Sender
BB	$\frac{9}{10}p_1 + \frac{29}{10}p_2 + \frac{29}{10}(1 - p_1 - p_2) = \frac{29}{10} - 2p_1$
BQ	$\frac{6}{5}p_1 + \frac{14}{5}p_2 + 3(1 - p_1 - p_2) = 3 - \frac{9}{5}p_1 - \frac{1}{5}p_2$
QQ	$\frac{21}{10}p_1 + \frac{1}{10}p_2 + \frac{21}{10}(1 - p_1 - p_2) = \frac{21}{10} - 2p_2$

Therefore the sender prefers

$$\begin{aligned} BB \text{ to } BQ &\iff p_1 \le p_2 - \tfrac{1}{2} \\ BB \text{ to } QQ &\iff p_1 \le p_2 + \tfrac{2}{5} \\ BQ \text{ to } QQ &\iff p_1 \le p_2 + \tfrac{1}{2} \end{aligned}$$

Therefore BB is a best response for the sender if and only if $p_1 \le p_2 - \frac{1}{2}$, BQ is a best response if and only if $p_2 - \frac{1}{2} \le p_1 \le p_2 + \frac{1}{2}$, and QQ is a best response if and only if $p_1 \ge p_2 + \frac{1}{2}$.

Next, let q_1 denote the probability with which the sender selects BB, q_2 the probability with which the sender selects BQ, and $q_3 = 1 - q_1 - q_2$ the probability with which the sender selects QQ. Then the receiver's expected payoff equals:

$$\begin{array}{ll} dn & -\frac{4}{5}q_1 - \frac{9}{10}q_2 \\ nd & \frac{1}{10}q_2 - \frac{4}{5}(1 - q_1 - q_2) = -\frac{4}{5} + \frac{9}{10}q_2 + \frac{4}{5}q_1 \\ nn & 0 \end{array}$$

Therefore the receiver prefers

$$\begin{aligned} dn \text{ to } nd &\iff q_1 \le \tfrac{1}{2} - \tfrac{9}{8}q_2 \\ dn \text{ to } nn &\iff q_1 = q_2 = 0 \\ nd \text{ to } nn &\iff q_1 \ge 1 - \tfrac{9}{8}q_2 \end{aligned}$$

Therefore dn is a best response for the receiver if and only if $q_1 = q_2 = 0$, nd is a best response if and only if $q_1 \ge 1 - \frac{9}{8}q_2$, and nn is a best response if and only if $q_1 \le 1 - \frac{9}{8}q_2$.

To determine the Nash equilibria, we consider three cases:

[1] BB is a best response to nd, BQ is a best response to nn, and QQ is a best response to dn. Also, dn is a best response to QQ, nd is a best response to BQ, and nn is a best response to QQ.

1. $q_1 = q_2 = 0$, i.e. the sender selects QQ with probability one. Then the receiver can respond with any mixture between dn and nn, i.e. $p_2 = 0$. Given this receiver response, QQ will be a best response if and only if $p_1 \geq \frac{1}{2}$.

2. $q_1 > 0$ or $q_2 > 0$. Then from the receiver's best response we obtain $p_1 = 0$. But when $p_1 = 0$, the sender will select QQ with probability zero, i.e. we must have $q_1 + q_2 = 1$.

   (a) $q_1 > 1 - \frac{9}{8}q_2$. Given that $q_1 + q_2 = 1$ this inequality is equivalent to $q_1 < 1$. When $q_1 > 1 - \frac{9}{8}q_2$, the receiver must use nd with probability 1, i.e. we have $p_2 = 1$. Now with $p_2 = 1$ the sender must select BB with probability 1, i.e. we have $q_1 = 1$, a contradiction.

   (b) $q_1 < 1 - \frac{9}{8}q_2$. Given that $q_1 + q_2 = 1$ this inequality is equivalent to $q_1 > 1$, a contradiction.

   (c) $q_1 = 1 - \frac{9}{8}q_2$. Given that $q_1 + q_2 = 1$ this equality is equivalent to $q_1 = 1$, i.e. the sender selects BB for sure. In this case, the receiver can mix between nd and nn, but we must have $p_1 = 0$. The mixture between nd and nn must be selected so that the sender prefers BB to BQ, i.e. $p_2 \geq \frac{1}{2}$. We conclude that it is a Nash equilibrium for the receiver to mix with between nd and nn, with nd being selected with probability of at least $\frac{1}{2}$, and for the sender to select BB with probability 1.

We summarize this discussion as follows:

Lemma 1 *In any Nash equilibrium the sender selects QB with probability zero, and the receiver selects dd with probability zero. Let q_1 denote the probability with which the sender selects BB, and q_2 the probability with which the sender selects BQ (with the remaining probability allocated to QQ). Also let p_1 denote the probability with which the receiver selects dn, and p_2 the probability with which the receiver selects nd (with the remaining probability allocated to nn). Then there are two components of Nash equilibria to the Beer Quiche game, given by: (i) $q_1 = q_2 = 0$, $p_1 \geq \frac{1}{2}$ and $p_2 = 0$; (ii) $q_1 = 1$, $p_2 \geq \frac{1}{2}$ and $p_1 = 0$.*

Consider the sender's expected payoff in the Nash equilibria associated with the first component, in which both sender types select Q. We have $u_1^* = \frac{21}{10}$. Meanwhile, the sender's expected utility from selecting the strategy BB in these equilibria equals $\frac{9}{10}p_1 + \frac{29}{10}(1 - p_1) = \frac{29}{10} - 2p_1 \leq \frac{19}{10}$, and his expected payoff from selecting BQ equals $\frac{6}{5}p_1 + 3(1-p_1) = 3 - \frac{9}{5}p_1 \leq \frac{21}{10}$. Thus only strategy BQ is ever a weak best response, and then only in the equilibrium where $p_1 = \frac{1}{2}$.

Consider the sender's expected payoff in the Nash equilibria associated with the second component, in which the sender selects BB and the receiver selects nd with probability $p_2 \geq \frac{1}{2}$ and nn with probability $1 - p_2$. We have $u_1^* = \frac{29}{10}$. Meanwhile, the sender's expected utility from selecting the strategy BQ in these equilibria equals $\frac{14}{5}p_2 + 3(1 - p_2) \leq \frac{29}{10}$, and his expected payoff from selecting QQ equals $\frac{1}{10}p_2 + \frac{21}{10}(1 - p_2) \leq \frac{21}{10}$. Thus only the strategy BQ is ever a weak best response, and then only in the equilibrium in which $p_2 = p_3 = \frac{1}{2}$.

Now let us consider matters from the receiver's perspective. Her equilibrium utility in the Nash equilibria associated with the first component of Nash equilibria in which the sender selects QQ equals $u_2^* = 0$. Therefore there are no weak best responses other than the equilibrium responses dn and nn.

In the Nash equilibria associated with the second component, in which the sender selects BB, the receiver's expected utility equals $u_2^* = 0$. Therefore there are no weak best responses other than the equilibrium responses nd and nn.

3 Stable Sets

To find the stable sets of this game, we make use of the following result, contained in Kohlberg-Mertens: Every stable set contains a stable set of the game in which a strategy that is weakly dominated or never a weak best response to any equilibrium in the set is eliminated. We restrict attention to stable sets contained in a single component of Nash equilibria.

For equilibria associated with the component in which the sender uses QQ, the strategies BB and nd are not a weak best response to any equilibrium in the component. The game in which these strategies have been eliminated is as follows:

	dn	nn
BQ	$\frac{6}{5}, -\frac{9}{10}$	$3, 0$
QQ	$\frac{21}{10}, 0$	$\frac{21}{10}, 0$

Since the strategy dn is weakly dominated, it must be used with minimal probability in any equilibrium of a pertubed game. But then the sender will select BQ with maximal probability, i.e. there is no equilibrium of the perturbed game near this component. We conclude that this component contains no stable set.

Next, consider equilibria associated with the component in which the sender uses BB. The strategies QQ and dn are not a weak best response to any equilibrium in the component, and after eliminating them, we obtain the following game:

	nd	nn
BB	$\frac{29}{10}, 0$	$\frac{29}{10}, 0$
BQ	$\frac{14}{5}, \frac{1}{10}$	$3, 0$

In this game, the strategy nn is weakly dominated, and hence will not be used in voluntarily any equilibrium of the perturbed game. But when the strategy nd is used with probability 1, the sender's best response is to select BB with maximal probability. The resulting equilibrium is near the component.

However, this does not end the analysis. To actually prove that this component is stable, we need to exhibit Nash equilibria for perturbations of the *full* game. This will be is second place where the Cho-Sobel condition comes in. Note in particular that we cannot merely turn the equilibrium of the reduced game into an equilibrium of the full game by having all strategies other than BB and nd being selected with minimal probability: if the sender trembles sufficiently more on QQ than on BQ then the receiver would want to use the strategy nn with maximal probability. And in that case the sender can profitably deviate by selecting BQ with maximal probability. A more intricate construction is needed. We illustrate this construction below.

3.1 Perturbations that can be Stabilized without Voluntary Contribution

Let $\varepsilon^s = (\varepsilon^s_{BB}, \varepsilon^s_{BQ}, \varepsilon^s_{QB}, \varepsilon^s_{QQ})$ be a vector of trembles for the sender, i.e. a vector satisfying $\varepsilon^s_i > 0$ for all $i \in S_s \equiv \{BB, BQ, QB, QQ\}$ and $\sum_{i \in S_s} \varepsilon^s_i < 1$. Similarly, let $\varepsilon^r = (\varepsilon^r_{dd}, \varepsilon^r_{dn}, \varepsilon^r_{nd}, \varepsilon^r_{nn})$ be a vector of trembles for the sender, i.e. a vector satisfying $\varepsilon^r_j > 0$ for all $j \in S_r \equiv \{dd, dn, nd, nn\}$ and $\sum_{j \in S_r} \varepsilon^r_j < 1$. Then the $(\varepsilon^s, \varepsilon^r)$ perturbed game is the (infinite) normal form game derived from the original game in which the sender is restricted to using completely mixed strategies $\sigma^s = (\sigma^s_{BB}, \sigma^s_{BQ}, \sigma^s_{QB}, \sigma^s_{QQ})$ satisfying $\sigma^s_i \geq \varepsilon^s_i$ for all $i \in S_s$, and the receiver is restricted to using completely mixed strategies satisfying $\sigma^r = (\sigma^r_{dd}, \sigma^r_{dn}, \sigma^r_{nd}, \sigma^r_{nn})$ satisfying $\sigma^r_j \geq \varepsilon^r_j$ for all $j \in S_r$.

We start by a simple observation : in any Nash equilibrium of any sufficiently small perturbed game near a component of equilibria of the unperturbed game, any strategy which is not a weak best response to any equilibrium in the component must be used with minimal probability.

Thus in any such Nash equilibrium near the component in which the sender uses BB with probability one we must have

$$\sigma^s_{QB} = \varepsilon^s_{QB},\ \sigma^s_{QQ} = \varepsilon^s_{QQ},\ \sigma^r_{dd} = \varepsilon^r_{dd},\ \sigma^r_{dn} = \varepsilon^r_{dn}.$$

First, we consider perturbations for which there are equilibria to the perturbed game near the component in which there are no voluntary contributions. That is, we consider equilibria in which $\sigma^s_{BQ} = \varepsilon^s_{BQ}$.

For such equilibria to exist, the sender's expected payoff from using the strategy BB must exceed his payoff from using BQ. The expected payoff from using BB equals $\frac{9}{10}(\varepsilon^r_{dd} + \varepsilon^r_{dn}) + \frac{29}{10}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn})$ for any allowed equilibrium strategy of the receiver. Meanwhile, the expected payoff from using BQ equals $\varepsilon^r_{dd} + \frac{6}{5}\varepsilon^r_{dn} + \frac{14}{5}\sigma^r_{nd} + 3\sigma^r_{nn}$, where $\sigma^r_{nd} + \sigma^r_{nn} = 1 - \varepsilon^r_{dd} - \varepsilon^r_{dn}$. Substituting in the constraint yields an expected payoff from using BQ equal to $\varepsilon^r_{dd} + \frac{6}{5}\varepsilon^r_{dn} + \frac{14}{5}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \sigma^r_{nn}) + 3\sigma^r_{nn}$, where $\varepsilon^r_{nn} \leq \sigma^r_{nn} \leq 1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \varepsilon^r_{nd}$. Since this payoff is increasing in σ^r_{nn}, BB will be an optimal strategy if and only if its expected payoff excceds the expected payoff from BQ when $\sigma^r_{nn} = \varepsilon^r_{nn}$. Thus it is necessary that $\frac{9}{10}(\varepsilon^r_{dd} + \varepsilon^r_{dn}) + \frac{29}{10}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn}) \geq \varepsilon^r_{dd} + \frac{6}{5}\varepsilon^r_{dn} + \frac{14}{5}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \varepsilon^r_{nn}) + 3\varepsilon^r_{nn}$. Straightforward algebra reduces this inequality to

$$\frac{1}{2} - \varepsilon^r_{dd} - \varepsilon^r_{nn} + 27\varepsilon^r_{dn} \geq 0$$

which always holds if the perturbation is sufficiently small.

The second condition for such equilibria to exist, is that the receiver's expected payoff from using the strategy nd must exceed her expected payoff from the strategy nn. The receiver's expected payoff from using the strategy nd equals $\frac{1}{10}\varepsilon^s_{BQ} - \frac{9}{10}\varepsilon^s_{QB} - \frac{4}{5}\varepsilon^s_{QQ}$. Meanwhile, her expected payoff from the strategy nn equals zero. Hence we must have

$$\varepsilon^s_{BQ} - 9\varepsilon^s_{QB} - 8\varepsilon^s_{QQ} \geq 0\,.$$

In particular, note that it is impossible for this inequality to be satisfied if $9\varepsilon^s_{QB} \geq \varepsilon^s_{BQ}$, as we predicted above.

We may summarize this discussion as follows:

Lemma 2 *(No Voluntary Constributions) For sufficiently small perturbations, the* $(\varepsilon^s, \varepsilon^r)$ *perturbed game has a Nash equilibrium without voluntary contributions near the component of the original game in which the sender selects BB with probability one if and only if* $\varepsilon^s_{BQ} - 9\varepsilon^s_{QB} - 8\varepsilon^s_{QQ} \geq 0$. *One such equilibrium is given by* $\sigma^s = (1 - \varepsilon^s_{BQ} - \varepsilon^s_{QB} - \varepsilon^s_{QQ},\ \varepsilon^s_{BQ},\ \varepsilon^s_{QB}, \varepsilon^s_{QQ})$ *and* $\sigma^r = (\varepsilon^r_{dd},\ \varepsilon^r_{dn},\ 1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \varepsilon^r_{nn}, \varepsilon^r_{nn})$.

3.2 Equilibria with Voluntary Contributions

When $\varepsilon^s_{BQ} - 9\varepsilon^s_{QB} - 8\varepsilon^s_{QQ} < 0$ only Nash equilibria with voluntary contribution can be near the component of the original game in which the sender selects BB with probability one. Here we document the construction of such equilibria.

For an equilibrium with voluntary contribution to exist, it must be the case that in the perturbed game the sender's expected payoff from using the strategy BB equals his expected payoff from using the strategy BQ. Since for sufficiently small perturbations in any equilibrium near the component the receiver mus use the strategies dd and dn with minimal probability, the sender's expected payoff form BB equals $\frac{9}{10}(\varepsilon^r_{dd} + \varepsilon^r_{dn}) + \frac{29}{10}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn})$. Meanwhile, as we computed above, his expected payoff from BQ equals $\varepsilon^r_{dd} + \frac{6}{5}\varepsilon^r_{dn} + \frac{14}{5}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \sigma^r_{nn}) + 3\sigma^r_{nn}$, where σ^r_{nn} must satisfy the inequalities $\varepsilon^r_{nn} \leq \sigma^r_{nn} \leq 1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \varepsilon^r_{nd}$. We showed above that at $\sigma^r_{nn} = \varepsilon^r_{nn}$ the inequality $\frac{9}{10}(\varepsilon^r_{dd} + \varepsilon^r_{dn}) + \frac{29}{10}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn}) > \varepsilon^r_{dd} + \frac{6}{5}\varepsilon^r_{dn} + \frac{14}{5}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \sigma^r_{nn}) + 3\sigma^r_{nn}$ holds when the perturbations are sufficiently small. Meanwhile, at $\sigma^r_{nn} \leq 1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \varepsilon^r_{nd}$, the inequality $\frac{9}{10}(\varepsilon^r_{dd} + \varepsilon^r_{dn}) + \frac{29}{10}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn}) < \varepsilon^r_{dd} + \frac{6}{5}\varepsilon^r_{dn} + \frac{14}{5}(1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \sigma^r_{nn}) + 3\sigma^r_{nn}$ holds when the perturbations are sufficiently small. Thus there exists a unique value of $\sigma^r_{nn} \in [\varepsilon^r_{nn}, 1 - \varepsilon^r_{dd} - \varepsilon^r_{dn} - \varepsilon^r_{nd}]$ such that the sender is indifferent between using BB and using BQ. This value can be calculated to be:

$$\sigma^r_{nn} = \frac{1}{2}(1 - 2\varepsilon^r_{dd} + 54\ \varepsilon^r_{dn})$$

Note in particular that $\sigma^r_{nn} \to \frac{1}{2}$ as the perturbation vanishes, i.e. we pick up the weak Nash equilibrium in the component! This is because it is the only equilibrium in the component of the unperturbed game for which the sender is indifferent between sending BB and BQ.

The second condition for such an equilibrium to exist is that the receiver is indifferent between sending nd and nn in the Nash equilibrium of the perturbed game. Since the receiver's expected payoff from using the strategy nd equals $\frac{1}{10}\sigma^s_{BQ} - \frac{9}{10}\varepsilon^s_{QB} - \frac{4}{5}\varepsilon^s_{QQ}$, and her expected payoff from using nn equals zero, this requires that

$$\sigma^s_{BQ} = 9\varepsilon^s_{QB} + 4\varepsilon^s_{QQ}.$$

We summarize this discussion as follows:

Lemma 3 *(Voluntary Constributions) For sufficiently small perturbations, the* $(\varepsilon^s, \varepsilon^r)$ *perturbed game has a Nash equilibrium with voluntary contributions near the component of the original game in which the sender selects BB with probability when* $\varepsilon^s_{BQ} - 9\varepsilon^s_{QB} - 8\varepsilon^s_{QQ} < 0$. *This equilibrium is given by* $\sigma^s = (1 - 10\varepsilon^s_{QB} - 5\varepsilon^s_{QQ},\ 9\varepsilon^s_{QB} + 4\varepsilon^s_{QQ},\ \varepsilon^s_{QB}, \varepsilon^s_{QQ})$ *and* $\sigma^r = (\varepsilon^r_{dd},\ \varepsilon^r_{dn},\ \frac{1}{2} - 28\varepsilon^r_{dn}, \frac{1}{2} - \varepsilon^r_{dd} + 27\ \varepsilon^r_{dn})$.