Comparative statics: The stochastic case

(Athey 2002)

$$U(x,\theta) = \int u(x,s)\ f(s,\theta) d\mu(s),$$

where $\theta \in R, x \in R^n, s \in R^m$, f is a density and μ is a product measure

- When is $x^*(\theta, B) =$arg max$_{x \in B} U(x,\theta) \nearrow$ in (θ, B) ?
- When is $U(x,\theta)$ log-spm in (x,θ)? Recall: log spm $\Longrightarrow$q-spm.
- Interpretation: $F : R^2 \to R$, $F \geq 0$, is log-spm if for
  $(x', y') > (x, y)$
  $F(x', y')F(x, y) \geq F(x', y)F(x, y')$ or
  $F(x', y')/F(x, y') \geq F(x', y)/F(x, y)$
  the relative returns $F(x', \cdot)/F(x, \cdot)$ are $\nearrow$ (or
  $F(x\cdot, y')/F(\cdot, y) \nearrow$).
  (as opposed to absolute returns for spm).

Log-spmineconomics:

- $D(p, \theta)$ is log-spm iff price elasticity is $\nearrow$ in θ (CLS 2).
- A marginal utility u' $(w + s)$ is log-spm in (w, s) iff u satisfies DARA or —$u''(w)/u'(w)$ is $\searrow$ in w.
- A c.d.f. $F(s; \theta)$ has a hazard rate $f(s; \theta)/[1 - F(s; \theta)]$ that is $\searrow$ in θ iff $F(s; \theta)$ is log-sbm.
- Random variables are affiliated if their joint density f is log-spm.
- If $suppF(s, \theta)$ constant, the MLR order requires $f(s, \theta)$ log-spm or $f(s, \theta')/f(s, \theta)$ $\nearrow$ in s for $\theta' > \theta$.
- Many known densities $f(\cdot)$ are log-concave, so $f(s + t)$ is log-sbm. Same for cdf's.

Lemma 1 $g(y,s)$ log-spm *in* $(y,s) \Longrightarrow \int g(y,s)\mathrm{d}\mu(s)$ log-spm *in* y.
Proof.

Lemma 2 (Ahlswed-Daykin79) *$h_1, h_2, h_3, h_4 : S \to R$ are ≥ 0 and s.t. $h_1(s)h_2(s^{'}) \leq h_3(s \vee s^{'})h_4(s \wedge s')$ for μ-almost all $s, s^{'} \in S$. Then*

$$\int h_1(s)d\mu(s) \cdot \int h_2(s)d\mu(s) \leq \int h_3(s)d\mu(s) \cdot \int h_4(s)d\mu(s).$$

Set $h_1(s) \triangleq g(y,s)$, $h_2(s) \triangleq g(y^{'},s)$, $h_3(s) \triangleq g(y \vee y^{'},s)$, $h_4(s) \triangleq g(y \wedge y^{'},s)$.

- In general, log-spm is not preserved by sums. Here it works because sums are of form $g(y,s) + g(y',s)$ where g is log-spm in (y,s)

Corollary 3 $x^*(\theta, B) \nearrow$ in θ if μ and f are log-spm (then U is log-spm.)

Lemma 4 *Let $A: R \to 2^S$ and $1_{A(t)}(s) = 1$if $s \in A(t)$ and 0 otherwise. $1_{A(t)}(s)$ is log-spm in (s, t) iff A_t is an ascending sublattice in t .*

Proof. Must show that for $t' > t$ and any s' , $s \in S$,

$$1_{A(t')}(s \vee s')1_{A(t)}(s \wedge s') \geq 1_{A(t')}(s)1_{A(t)}(s')$$

or that RHS $= 1 \implies$ LHS $= 1$.

Corollary 5 *$f(s,\theta)$ log-spm in (s,θ)*
i.e. satisfies MLR
$\Longrightarrow F(s,\theta) = \int 1_{[a,b]}(s) f(s,\theta) ds$ log-spm in (s,θ),
i.e. satisfies M.Prob.R.,
which $\Longrightarrow$ FOSD

Proof. $1_{[a,b]}(s)$ is log-spm in (a,b,s). Use Lemma 4.

Corollary 6 *$F(s,\theta)$ log-spm in $(s,\theta) \Longrightarrow \int_{-\infty}^{a} F(s,\theta) ds$ is log-spm in(a,θ), which $\Longrightarrow$ SOSD.*

Applications to ratio orders (e.g. DARA):

- Let $g, h : R \to R_+$ and $u : \{0,1\} \times R \to R_+$ be s.t. $u(x,s) = g(s)$ if $x = 1$ and $u(x,s) = h(s)$ if $x = 0$. Then, if $h > 0$, u is log-spm iff $g(s)/h(s)$ is $\nearrow$in s since with $s^{'} > s$,

  $$u(1,s^{'})u(0,s) \geq u(0,s^{'})u(1,s) \Leftrightarrow g(s^{'})h(s) \geq g(s)h(s^{'})$$

  By Lemma 1, if f is log-spm, $g(s)/h(s)$ is $\nearrow$ in s $\Longrightarrow U(x,\theta) = \int u(x,\theta)f(s,\theta)d\mu(s)$ is log-spm in (x,θ) or $\frac{g(s)f(s,\theta)ds}{h(s)f(s,\theta)ds}$ is in θ, since with $\theta' > \theta$,

  $$U(1,\theta')U(0,\theta) \geq U(0,\theta')U(1,\theta) \iff$$
  $$\int g(s)f(s,\theta^{'})d\mu \int h(s)f(s,\theta)d\mu \geq$$
  $$\int h(s)f(s,\theta')d\mu \int g(s)f(s,\theta)d\mu.$$

- Preservation of risk aversion orders by expectations. Assume DARA (or $u''(w)/u'(w)$ in w or $u'(w+s)$ log-spm) and f log-spm. Then if $U(w) = \int u(w+s)f(s,\theta)ds$, we have $U'(w+t) = \int u'(w+t+s)f(s,\theta)ds$ and U' is log-spm or U''/U' is $\nearrow$.
- MLR shifts and DARA: Assume DARA (or $u^{''}(w)/u^{'}(w)$ in w or $u^{'}(w+s)$ log-spm) and f log-spm. Then $\frac{U_{ww}(w,\theta)}{U_w(w,\theta)} = \frac{u(w+s)f(s,\theta)ds}{u(w+s)f(s,\theta)ds}$ is $\nearrow$ in θ
- Same arguments for prudence: $-u^{'''}/u^{''}$

- **Log–spm Bayesian games** (Athey, Econ' 2001):

  Player i has type $t_i \in T_i$ and uses strategy $s_i(\cdot) : T_i \to A_i \subset R$.
  Let $T = \Pi_i T_i$ and $h > 0$ be the joint density over types, and $h_i(\cdot/t_i)$ be the conditional density of other players' types.
  The utility of i is $v_i : A \times T \to R$.
  Expected payoff is

  $$V_i(x_i, t_i) = \int v_i(x_i, s_{-i}(t-i), t) h_{-i}(t_{-i}/t_i) dt_{-i}$$

Theorem 7

(a)*Let the types be affiliated* ($i.e.$ $h(t)$ is *log-spm in* t).
Then $s_i(t_i) = \arg\max V_i(x_i, t_i)$ is $\nearrow$ if $s_{-i}(t-i)$ *is* $\nearrow$.

(b) *There exists a PSNE if* v_i *is cont*. *in a and* A_i *is compact convex* (*in* R), *or if* A_i *is finite*.

Applications:

- Auctions with affiliated values (Milgrom-Weber, Econ'1982)
- Bertrand competition with linear costs and incomplete information
  (with affiliated costs.)
- **Related: Stochastic games** (Amir, GEB 1996).