NTU IO (I) : Classnote 06 *Mechanism Design -Application: durable goods monopoly*

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The market is populated by a continuum of infinitely-lived consumers, indexed by $q \in I = [0, 1]$. All consumers are risk neutral and have the same discount rate r. Each consumer wishes to possess at most one unit of the durable good. Let $f(q)$ denote consumer q's willingness to pay for the privilege of a one-time opportunity of acquiring one unit of the durable good.

Suppose that the monopolist trades with consumer q at the price $\pi(q)$ at time $t(q)$.

q's surplus:

$$U(q) = e^{-rt(q)} [f(q) - \pi(q)] \tag{1}$$

Probability of trade (discount on time of trade):

$$X(q) = e^{-rt(q)} \in [0, 1]$$

Discount payment

$$T(q) = e^{-rt(q)} \pi(q)$$

Therefore, we can rewrite (1) as

$$U(q) = f(q) X(q) - T(q)$$

A mechanism design should satisfies incentive-compatibility constraints. Hence, we have

$$\begin{cases} f(q)X(q) - T(q) \geq f(q)X(q') - T(q') \\ f(q')X(q') - T(q') \geq f(q')X(q) - T(q) \end{cases}$$

Rewriting the above inequalities, we obtain

$$\begin{cases} T(q) - T(q') \leq f(q)[X(q) - X(q')] \\ T(q) - T(q') \geq f(q')[X(q) - X(q')] \end{cases}$$

Therefore, we have

$$\Longrightarrow [f(q') - f(q)][X(q) - X(q')] \leq 0$$

Monotonicity constraints:

First if $q' \geq q$ and $f(q') < f(q)$ (strictly decreasing demand curve)

(1) **then** $X(q) \geq X(q')$

When $f(q) = f(q')$(not strictly decreasing)

just relabel the q. such that $X(q)$ is weakly decreasing.

Secondly, $T(q) - T(q') \geq f(q')[X(q) - X(q')] \geq 0$

Hence in any incentive-compatibe bargaining mechanism. (ICBM)

(2)$T(q)$ **is weakly decreasing.**

Consumer with lower value will trade later at a lower discounted price.

Finally, let $\hat{U}(q, q') = f(q) X(q') - T(q')$. Hence, IC constraint requires that

$$U(q) = \max_{q' \in [0,1]} \hat{U}(q, q')$$

Envelope Theorem then implies that $U'(q) = \hat{U}_q(q, q) = f'(q) X(q)$.

Therefore, we have

$$\begin{aligned} U(1) - U(q) &= \int_q^1 U'(z)\, dz = \int_q^1 f'(z) X(z)\, dz \\ &= f(z) X(z)|_q^1 - \int_q^1 f(z)\, dX(z) \end{aligned}$$

$$f(1) X(1) - T(1) - f(q) X(q) + T(q) = f(1) X(1) - f(q) X(q) - \int_q^1 f(z)\, dX(z)$$

(3)equivalent

$$T(q) = T(1) - \int_q^1 f(z)\ \mathrm{d}X(z) \qquad \text{Stieltjes integral} \tag{2}$$

$$f(z)\Delta X(z) \text{ at a jump point of } X(z)$$

$$\text{or } f(q)[X(q) - X(q')] \geq T(q) - T(q') \geq f(q')[X(q) - X(q')]$$

a jump in $X(q) \Rightarrow$ a jump in $T(q)$

Theorem: $X : [0,1] \to [0,1]$ and $T : [0,1] \to R_+$ is a ICBM if and only if (1)-(3) holds.

Let $f(1) = MC$ then we have $t(1) = \infty$ (no gap case) then $X(1) = T(1) = 0$ and $T(q) = -\int_q^1 f(z)\mathrm{d}X(z)$

Show $\forall T(q)$ the net present value of durable = sum of discounted future rentals.

Let $MC = c$

$$
\begin{aligned}
\text{monopolist's profits} &= \int_0^1 [T(q) - cX(q)]\ \mathrm{d}q \\
&= \int_0^1 [-\int_q^1 f(z)\ \mathrm{d}X(z) - cX(q)]\ \mathrm{d}q \\
&= \int_0^1 \{-\int_q^1 [f(z) - c]\ \mathrm{d}X(z)\}\ \mathrm{d}q \\
&= \int_0^1 \int_0^z [f(z) - c]\ \mathrm{d}q\ \mathrm{d}(1 - X(z)) \\
&= \int_0^1 (f(z) - c)\ z\ \mathrm{d}(1 - X(z))
\end{aligned}
$$

Suppose $(f(z) - c)z$ has a unique maximizer z^*

$\Rightarrow$put all mass at $z = z^*$

$\Rightarrow$profit= $[f(z^*) - c]z^*$ same as max static profits function.

Thus, the optimal selling mechanism is to set the quantity at the maximum static profit, then never sell after.

No intertemporal price discrimination.

The optimal strategy for the monopoly renter:

(1)renting

Rental demand curve $\frac{1-\delta}{r}F(q)$

q^* to maximize $q^*F(q^*)$

p^* to maximize $p^*F^{-1}(p^*)$

(2)Sales profits $\leq$ rental profits

renting strategies $\supset$ sale strategies

$\because$for any sale prices $...p_t \;\; p_{t+1}...$, we can compute the associated rental prices by

$R_{t+1} = p_t - \delta p_{t+1}$

Hence, any sales policy can be duplicated by a rental policy.

(3)the optimal rental policy can be duplicated by a sales policy

Sell X^* at period 0. open loop: X_t is a function of t only.