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The Nash-Equilibrium Correspondence Has a Closed Graph

$G_1(\lambda)$:

L	R
$1+\lambda$	$1-\lambda$

Let x denote the probability that the decision maker plays L, and consider the optimal x for each λ. This defines the Nash-equilibrium correspondence for this one-player game.

$$x^*(\lambda) \in E(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda < 0 \\ [0,1] & \text{if} \quad \lambda = 0 \\ 1 & \text{if} \quad \lambda > 0 \end{cases}.$$

Two Important Properties:

Proposition 1 *The graph of the Nash correspondence has a closed graph (is upper hemi-continous), i.e., for any sequence (λ^n, x^n) belonging to the graph of the correspondence and converging to some (λ, x), the limit (λ, x) belongs to the graph of correspondence.*

Proposition 2 *The correspondence may not be "lower hemi-continuous". That is, there may exist (λ, x) belonging to the graph of the correspondence and a sequence $\lambda^n \to \lambda$ such that there exists no x^n such that (λ^n, x^n) belongs to the graph of the correspondence and $x^n \to x$. Here, take $\lambda = 0$ and $x \in (0,1)$.*

$G_2(\lambda)$:

	L	R
U	$1,1$	$0,0$
D	$0,0$	$\lambda, 2$

$G_3(\lambda)$:

	L	R
U	$1,1$	$0,0$
D	$0,0$	λ, λ

Let p denote the probability of U and q denote the probability of L,
the Nash equilibrium correspondence for $G_2(\lambda)$:

$$(p^*(\lambda), q^*(\lambda)) \in E(\lambda) = \begin{cases} \{(1,1)\} & \text{if} \quad \lambda < 0 \\ \{(1,1), \left(x, \frac{\lambda}{1+\lambda}\right), (0,0) : x \in [0, \frac{2}{3}]\} & \text{if} \quad \lambda = 0 \\ \{(1,1), \left(\frac{2}{3}, \frac{\lambda}{1+\lambda}\right), (0,0) & \text{if} \quad \lambda > 0 \end{cases}$$

the Nash equilibrium correspondence for $G_3(\lambda)$:

$$(p^*(\lambda), q^*(\lambda)) \in E(\lambda) = \begin{cases} \{(1,1)\} & \text{if} \quad \lambda < 0 \\ \{(1,1), \left(\frac{\lambda}{1+\lambda}, \frac{\lambda}{1+\lambda}\right), (0,0) & \text{if} \quad \lambda \geq 0 \end{cases}$$

One additional property.

Proposition 3 *(Wilson's (1971) Oddness Theorem) Almost all finite games have a finite and odd number of equilibria.*

Kohlberg and Mertens (Econometrica 1986)

Proposition 4 *For generic extensive-form games, the set of Nash equilibrium outcomes is finite and all Nash equilibrium within a given connected component induce the same outcome.*

Proposition 5 *Every game has at least one stable component.*

Properties on Sequential Equilibrium

Definition 6 *An assessment is a pair* (b, μ) *where* b *is a behavior strategy combination and* μ *is a system of beliefs.*

Definition 7 *An assessment* (b, μ) *is consistent if there exists a sequence* $\{b^\varepsilon, \mu^\varepsilon\}_{\varepsilon \downarrow 0}$ *where* b^ε *is a completely mixed behavior strategy combination and* μ^ε *is the system of beliefs generated by* b^ε *such that*

$$\lim_{\varepsilon \downarrow 0} (b^\varepsilon, \mu^\varepsilon) = (b, \mu)$$

Definition 8 *A sequential equilibrium is a consistent assessment* (b, μ) *for which* b *is a sequential best reply against* (b, μ)*.*

Proposition 9 *Every sequential equilibrium is subgame perfect.*

Definition 10 *Let* Γ *be an extensive form game. If* η *is a mapping which assigns to every choice in* Γ *a positive number* η_c *such that* $\sum_{c \in C_u} \eta_c < 1$ *for every information set* u*, then the perturbed game* (Γ, η) *is the extensive form game with the same structure as* Γ*, but in which every player* i *is only allowed to use behavior strategies* b_i *which satisfy* $b_{iu}(c) \geq \eta_c$ *for all* $u \in U_i$ *and* $c \in C_u$*.*

Definition 11 b *is a perfect equilibrium of* Γ *if* b *is a limit point of a sequence* $\{b(\eta)\}_{\eta \downarrow 0}$ *where* $b(\eta)$ *is an equilibrium of* (Γ, η)*.*

We will use the term "(game) tree" for the extensive form of a game with perfect recall (i.e., where every player remembers whatever he knew previously, including his past actions).

The agent normal form (Selten) of a tree is the normal form of the game between agents, obtained by letting each information set be manned by a different agent, and by giving any agent of the same player that player's payoff. A behavioral strategy of a player in a tree is a list of (mixed) strategies, one for each of his agents. Kuhn (1953) has shown that every mixed strategy of a player in a tree is equivalent to some behavioral strategy, in the sense that both give the same probability distribution on the endpoints whatever be the strategies of all opponents.

A sequential equilibrium (Kreps-Wilson) of an n-player tree is an n-tuple of behavioral strategies which is the limit of a sequence (σ_m) of completely mixed (i.e., strictly positive) behavioral strategies, such that every agent maximizes his expected payoff given the strategies of all other agents and given the limiting conditional probability distribution on his information set implied by (σ_m).

An ε-perfect equilibrium of a normal form game (Selten) is a completely mixed strategy vector, such that any pure strategy which is not a best reply has weight less than E.

An ε-proper equilibrium of a normal form game (Myerson) is a completely mixed strategy vector, such that whenever some pure strategy s_1, is a worse reply than some other pure strategy s_2,, the weight on s_1, is smaller than ε times the weight on s_2,.

A perfect (proper) equilibrium of a normal form game is a limit ($\varepsilon \to 0$) of ε-perfect (proper) eqilibria.

A perfect (proper) equilibrium of a tree is a perfect (proper) equilibrium of its agent normal form.

It is evident that "proper" is a stronger requirement than "perfect." It is also easy to verify that a perfect equilibrium of a tree is sequential (Kreps-Wilson).

Existence theorems have been proved for all the above concepts (Kreps-Wilson, Myerson, Selten).

Example 1: A Perfect equilibrium of Γ need not be a perfect equilibrium of $N(\Gamma)$ (the normal form of Γ).

Example 2: A Perfect equilibrium of $N(\Gamma)$ is not necessarily a perfect euqilibrium of *Gamma*.

Example 3: An unreasonable subgame perfect equilibrium could be eliminated by perfect requirements

Example 3-1: An unreasonable subgame perfect equilibrium could pass perfect and sequential requirements.

Example 4: A sequential equilibrium need not be a perfect equilibrium

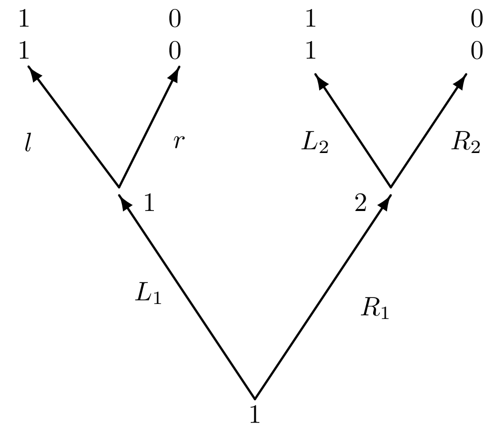
Proposition 12 *(Kreps and Wilson [1982])*

1. *Every perfect equilibrium is sequential.*
2. *For almost all extensive form games, almost all sequential equilibria are perfect.*
3. *For almost all games, the set of sequential equilibrium outcomes (i.e. the set of probability distributions over the endpoints resulting from sequential equilibria) coincides with the set of perfect equilibrium outcomes.*

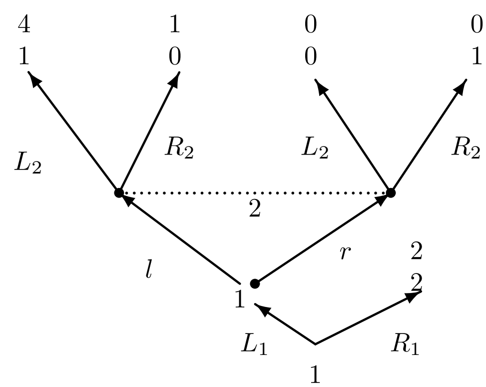
Proposition 13 *(Van Damme (1984), Kohlberg and Mertens (1986))*

1. *If s is a proper equilibrium of $N(\Gamma)$, then $\boldsymbol{P}^s$ is a sequential equilibrium outcome in Γ.*
2. *If b is a limit point of a sequence $\{b^\varepsilon\}_{\varepsilon\downarrow 0}$ where b^ε is induced by an ε-proper equilibrium s^ε of $N(\Gamma)$, then b is a sequential equilibrium of Γ.*

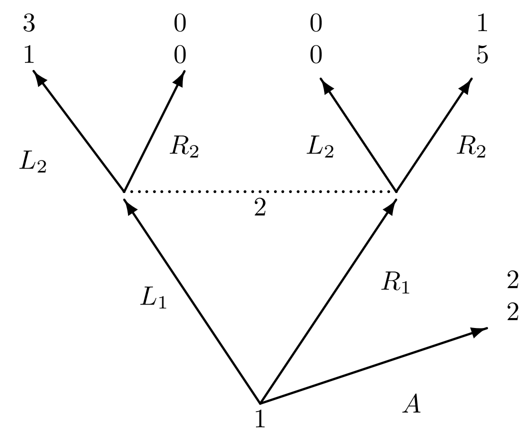
Example 1



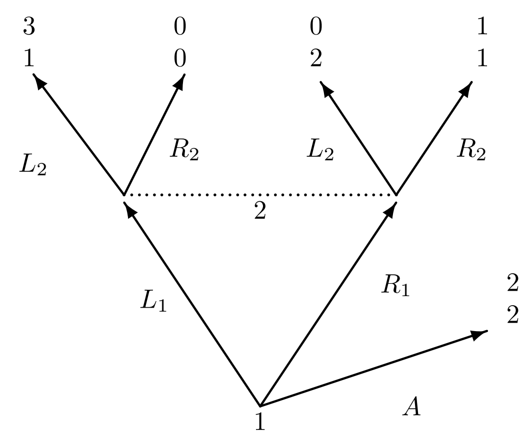
Example 2



Example 3



Example 3-1



Example 4