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NTU IO (I) : *Auction Theory and Mechanism Design*

1 Double Auction (Chatterjee and Samuelson 1983)

Bayesian Nash equilibria:

Game structure:

N : set of players

S_i : action space for i

Θ_i : set of types for i.

F : probability measure on $\Theta = \prod_{i \in N} \Theta_i$, $\theta \in \Theta$. "Prior"

$\pi_i(s_i, s_{-i}, \theta_i, \theta_{-i})$ payoff function

strategies: $s_i : \Theta_i \to S_i$

Definition 1 *$s_i^*(\theta_i), i \in N$, is a BNE if for $\forall \theta_i, \forall i \in N$*

$$s_i^*(\theta_i) \in \arg\max_{s_i \in S_i} \int \pi_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) F(\theta_{-i}|\theta_i) d\theta_{-i}$$

$N = 2$
b : buyer
s : seller
v : buyer's willingness to pay.
c : seller's cost, (contiuous types)
v, $c\tilde{}$ $[0, 1]$ uniformly
p_b and p_s are buyer's and seller's bids, respectively.

$$\pi_b(p_b, p_s, v, c) = \begin{cases} v - \frac{p_b + p_s}{2} & \text{if } p_b \geq p_s \\ 0 & \text{o.w.} \end{cases}$$

$$\pi_s(p_b, p_s, v, c) = \begin{cases} \frac{p_b + p_s}{2} - c & \text{if } p_b \geq p_s \\ 0 & \text{o.w.} \end{cases}$$

Note : If v, c are public information (no uncertainty) then this is a Nash demand Game. Any $p_b = p_s = p \in [c, v]$ is a N.E. and efficiency is attainable. However, if we have asymetric information, is efficiency attainable?

If a pure strategy $(p_b(v), p_s(c))$ is BNE then

$p_b(v)$ solves

$$\max_{p_b}\left[v-\frac{p_b+E\left(p_s(c)\,|p_s(c)\le p_b\right)}{2}\right]prob\left(p_s(c)\le p_b\right)$$

$p_s(c)$ solves

$$\max_{p_s}\left[\frac{p_s+E\left(p_b(v)\,|p_b(v)\ge p_s\right)}{2}-c\right]prob(p_b(v)\ge p_s)$$

Case 1: consider the following strategies

$p_b(v)=\begin{cases} x & \text{if } v\ge x \\ 0 & \text{o.w.}\end{cases}$ and $p_s(c)=\begin{cases} x & \text{if } c\le x \\ 1 & \text{o.w.}\end{cases}$ is a BNE.

Case 2: assume using linear strategies:

$$\begin{cases} p_b(v)=\alpha_b+\beta_b v \\ p_s(c)=\alpha_s+\beta_s c\end{cases} \tag{1}$$

where $\beta_s,\beta_b>0$

i.e.p_b ~unif $[\alpha_b,\alpha_b+\beta_b]$ and p_s ~unif $[\alpha_s,\alpha_s+\beta_s]$.

By the definition of BNE, we have (p_b^*,p_s^*) solves

$$\begin{cases} \max_{p_b}\left(v-\frac{1}{2}\left(p_b+\frac{\alpha_s+p_b}{2}\right)\right)\frac{p_b-\alpha_s}{\beta_s} \\ \max_{p_s}\left(\frac{1}{2}\left(p_s+\frac{p_s+\alpha_b+\beta_b}{2}\right)-c\right)\frac{\alpha_b+\beta_b-p_s}{\beta_b}\end{cases}$$

F.O.C.

$$\begin{cases} p_b=\frac{2}{3}v+\frac{1}{3}\alpha_s \\ p_s=\frac{2}{3}c+\frac{1}{3}\left(\alpha_b+\beta_b\right)\end{cases}$$

Comparing with (1), we have $\beta_b=\frac{2}{3}$,$\beta_s=\frac{2}{3}$, $\alpha_b=\frac{1}{12}$, $\alpha_s=\frac{1}{4}$

$$\begin{cases} p_b=\frac{1}{12}+\frac{2}{3}v \\ p_s=\frac{1}{4}+\frac{2}{3}c\end{cases}$$

Note:

- At $c=1$, $p_s=\frac{11}{12}<c$: The seller bids less than his own cost. Hence, the probability of trade at $c=1$ should be 0.
- At $v=0$, $p_b=\frac{1}{12}>v$: The buyer bids more than her own valuation. Hence, the probability of trade at $v=0$ should be 0.

Trade only happens when $\frac{1}{12}+\frac{2}{3}v\ge\frac{1}{4}+\frac{2}{3}c$, i.e., $v>c+\frac{1}{4}$.

Therefore efficient trade does not occur.

Q: Could we find a mechanism let trade occur for all $v>c$? No Way.

In fact : the second mechanism is the best mechanism in double auction game.

2 *Mechanism Design I*

Suppose that there are $I+1$ players:

- a principal (player 0) with no private information
- I agents $(i=1,\ldots,I)$ with types $\theta=(\theta_1,\ldots,\theta_I)$ in some set Θ.

Step 1: the principal designs a "mechanism," or "contract," or "incentive scheme."

Step 2: the agents simultaneously accept or reject the mechanism.

Step 3: the agents who accept the mechanism play the game specified by the mechanism. (send message $m(\theta)\in M$)

Principal chooses an allocation $y(m)=\{x(m),t(m)\}$.

- a decision $x\in X$, where X is a compact, convex and nonempty set
- a transfer $t=(t_1,\ldots,t_I)$ from the principal to each agent

Player i $(i=0,\ldots,I)$ has a von Neumann-Morgenstern utility $u_i(y,\theta)$. u_i $(i=1,\ldots,I)$ is increasing in t_i. u_0 is decreasing in each t_i. These functions are twice continuously differentiable.

- Agents: $U_i(\theta_i)=E_{\theta_{-i}}\left[u_i\left(y\left(\theta_i,\theta_{-i}\right),\theta_i,\theta_{-i}\right)|\theta_i\right]$
- Principal: $E_\theta u_0\left(y^*(\theta),\theta\right)$

Revelation Principle: The principal can content herself with "direct" mechanism, in which the message spaces are the type spaces, all agents accept the mechanism in step 2 regardless of their types, and the agents simultaneously and truthfully announce their types in step 3. (Gibbard (1973), Green and Laffont (1977), Dasgupta et al (1979) and Myerson (1979)).

Therefore we consider $y(\theta)$ instead of $y(m)$.

Goal: Find $y^*(\theta)$ such that y^* solves the principal's maximization problem

$$\max_y E_\theta u_0\left(y(\theta),\theta\right)$$

subject to

- IC constraints (Truth telling: Each agent's optimal choice is to report his own type θ_i)

$$u_i\left(y\left(\theta_i,\theta_{-i}\right),\theta\right)\geq u_i\left(y\left(\hat{\theta}_i,\theta_{-i}\right),\theta\right)\text{ for }\left(\theta_i,\hat{\theta}_i\right)\in[\underline{\theta},\bar{\theta}]\times[\underline{\theta},\bar{\theta}],\text{ and }i=1,\ldots,I$$

- IR constraints (participation constraint)

$$u_i\left(y\left(\theta_i,\theta_{-i}\right),\theta\right)\geq\underline{u}_i\text{ for all }\theta_i,\ i=1,\ldots,I.$$

Examples of Mechanism Design:

Seller-buyer example: Myerson and Satterthwaite (JET, 1983):

Suppose that the seller's cost and the buyer's valuation have differentiable, strictly positive densities on $[\underline{c}, \bar{c}]$ and $[\underline{v}, \bar{v}]$, that there is a positive probability of gains from trade $(c < \bar{v})$, and that there is a positive probability of no gains from trade $(\bar{c} > \underline{v})$. Then there is no efficient trading outcome that satisfies individual rationality, incentive compatibility and budget balance.

Model: The seller can supply one unit of a good at cost c drawn from distribution $F_1(\cdot)$ with differentiable, strictly positive density $f_1(\cdot)$ on $[\underline{c}, \bar{c}]$. The buyer has unit demand and valuation v drawn from distribution $F_2(\cdot)$ on $[\underline{v}, \bar{v}]$ with differentiable, strictly positive density $f_2(\cdot)$.

Principal: the social planner

agents: $I = 2$, seller and buyer

$x(c, v) \in [0, 1]$ the probability of trade

$t(c, v)$ the transfer from buyer to the seller (so $t_1 \equiv t$ and $t_2 \equiv -t$)

To find the optimal mechanism $y = \{x(c, v), t(c, v)\}$, let us define the followings:

$X_1(c) \equiv E_v[x(c, v)]$

$X_2(v) \equiv E_c[x(c, v)]$

$T_1(c) \equiv E_v[t(c, v)]$

$T_2(v) \equiv -E_c[t(c, v)]$

$U_1(c) \equiv T_1(c) - cX_1(c)$

$U_2(v) \equiv vX_2(v) + T_2(v)$

Note that the IC condition requires that $c \in \arg\max_{c'} T_1(c') - cX_1(c')$. Hence, envelope theorem implies that

$$\frac{dU_1(c)}{dc} = -X_1(c)$$

Therefoer, IR and IC conditions can be rewritten as

$$U_1(c) = U_1(\bar{c}) + \int_c^{\bar{c}} X_1(\gamma)\, \mathrm{d}\gamma$$
$$U_2(v) = U_2(\underline{v}) + \int_{\underline{v}}^{v} X_2(\nu)\, \mathrm{d}\nu$$

Substituting for $U_1(c)$ and $U_2(v)$ and adding up the above two equations yields

$$T_1(c) + T_2(v) = cX_1(c) - vX_2(v) + U_1(\bar{c}) + U_2(\underline{v}) + \int_c^{\bar{c}} X_1(\gamma)\, \mathrm{d}\gamma + \int_{\underline{v}}^{v} X_2(\nu)\, \mathrm{d}\nu$$

But budget balance $(t_1(c, v) + t_2(c, v) = 0)$ implies that

$$E_c T_1(c) + E_v T_2(v) = 0$$

Therefore

$$0 = \int_{\underline{c}}^{\bar{c}} \left(cX_1(c) + \int_{c}^{\bar{c}} X_1(\gamma)\,\mathrm{d}\gamma \right) f_1(c)\,\mathrm{d}c + U_1(\bar{c})$$
$$+ \int_{\underline{v}}^{\bar{v}} \left(\int_{\underline{v}}^{v} X_2(v)\,\mathrm{d}v - vX_2(v) \right) f_2(v)\,\mathrm{d}v + U_2(\underline{v})$$

$$U_1(\bar{c}) + U_2(\underline{v}) = -\int_{\underline{c}}^{\bar{c}} \left(c + \frac{F_1(c)}{f_1(c)} \right) X_1(c)\, f_1(c)\,\mathrm{d}c$$
$$+ \int_{\underline{v}}^{\bar{v}} \left(v - \frac{1 - F_2(c)}{f_2(v)} \right) X_2(v)\, f_2(v)\,\mathrm{d}v$$

$$U_1(\bar{c}) + U_2(\underline{v})$$
$$= \int_{\underline{c}}^{\bar{c}} \left(\int_{\underline{v}}^{\bar{v}} \left(v - \frac{1 - F_2(v)}{f_2(v)} \right) - \left(c + \frac{F_1(c)}{f_1(c)} \right) \right) x(c, v)\, f_1(c)\, f_2(v)\,\mathrm{d}c\mathrm{d}v \tag{2}$$

Consider the example in note 1: v, c are uniformly distributed on $[0, 1]$. Then (1) becomes

$$0 \le \int_0^1 \int_0^1 (2v - 1 - 2c)\, x(c, v)\,\mathrm{d}c\mathrm{d}v$$
$$= 2 \int_0^1 \int_0^1 \left(v - c - \frac{1}{2} \right) x(c, v)\,\mathrm{d}c\mathrm{d}v$$

$$\frac{\int_0^1 \int_0^1 (v - c)\, x(c, v)\,\mathrm{d}c\mathrm{d}v}{\int_0^1 \int_0^1 x(c, v)\,\mathrm{d}c\mathrm{d}v} \ge \frac{1}{2}$$

Hence, conditional on the individuals reaching an agreement to trade, the expected difference in their valuations must be at least $\frac{1}{2}$.

Note: the linear strategies in the double auction imply that $x(c, v) = 1$ iff $v - c \ge \frac{1}{4}$ and $x(c, v) = 0$ otherwise. Hence, the density on the trading area is $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{32}$. Conditional on the individuals reaching an agreement to trade, the expected difference in their valuations is $\int_{\frac{1}{4}}^{1} \int_0^{v - \frac{1}{4}} \frac{32}{9} (v - c)\mathrm{d}c\mathrm{d}v = \frac{1}{2}$ which satisfying the requirement. In fact, this is the second-best mechanism.

However, the ex post efficiency requires that conditional on the buyer's valuation being higher than the seller's, the expected differences $v - c$ would be only

$$\int_0^1 \int_0^v 2(v - c)\,\mathrm{d}c\mathrm{d}v = \frac{1}{3}$$

Hence, the smallest lump-sum subsidy required from an outside party to create a Bayesian incentive-compatible mechanism which is both ex post efficient and individually rational is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.