NTU Homework 4_solution

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1. (50 points) Similar to the nonlinear-prices example in class, a monopolist offers a menu of bundles $\{q, T\}$ to consumers, where q is the quantity sold and T is the fixed fee. Consumers with type θ receive utility $\theta\sqrt{q} - T(q)$ if they pruchase a quantity q and 0 otherwise. Suppose (q_1, T_1) is directed at type-θ_1 consumers (in proportion λ), and (q_2, T_2) is directed at type-θ_2 consumers(in proportion $1-\lambda$). The unit cost of producing the good is 1. Suppose $\theta_1 = 1$ and $\theta_2 = 4$.

   (a) (10 points) Is it possible that the monopolist's best strategy is to sell to type-θ_1 only?

   (b) (10 points) If the monopolist only sells to type-θ_2 consumers, what is the optimal budle (q_2^*, T_2^*)?

   (c) (10 points) If the monopolist sells to both types, what are the optimal bundles (q_1^*, T_1^*) and (q_2^*, T_2^*)?

   (d) (10 points) What is monopolist optimal decision?

   (e) (10 points) Suppose θ is uniformly distributed over $[1, 4]$. Find the optimal nonlinear price (q, T).

Ans:

(a) IR constrants for consumers:

$$\text{IR}_1:\ \theta_1\sqrt{q} - T \geq 0$$

$$\text{IR}_2:\ \theta_2\sqrt{q} - T \geq 0$$

Since $\text{IR}_2 > \text{IR}_1$, it is impossible for the monopolist to sell the goods to type-θ_1 consumers only.

□

The Monopolist's profit maximization problem:

$$\max_{\{q,T\}} E\pi = \lambda[T_1 - cq_1] + (1-\lambda)[T_2 - cq_2]$$

subject to:

$$\text{IR}_1:\ \theta_1\sqrt{q_1} - T_1 \geq 0$$

$$\text{IR}_2:\ \theta_2\sqrt{q_2} - T_2 \geq 0$$

$$\text{IC}_1:\ \theta_1\sqrt{q_1} - T_1 \geq \theta_1\sqrt{q_2} - T_2$$

$$\text{IC}_2:\ \theta_2\sqrt{q_2} - T_2 \geq \theta_2\sqrt{q_1} - T_1$$

There are only IR_1 and IC_2 binding.

Proof. If IR_1 is non-binding, we have $\theta_2\sqrt{q_2} - T_2 \geq \theta_2\sqrt{q_1} - T_1 > \theta_1\sqrt{q_1} - T_1 > 0$. Contradiction could occur when we rise T_1 and T_2 simultaneously. According to the discussion above, we can conclude that IR_1 is binding. If IR_1 is binding, then $\theta_2\sqrt{q_2} - T_2 \geq \theta_2\sqrt{q_1} - T_1 > \theta_1\sqrt{q_1} - T_1 > 0$. That is, IR_2 is non-binding. If IC_2 is non-binding, we have $\theta_2\sqrt{q_2} - T_2 > \theta_2\sqrt{q_1} - T_1 > 0$. Contradction could occur when rising T_2 can make profit. Now we can conclude that IC_2 is binding. If IC_2 is binding, we have $\theta_1[(\sqrt{q_1} - \sqrt{q_2})/(T_2 - T_1)] < \theta_2[(\sqrt{q_1} - \sqrt{q_2})/(T_2 - T_1)] = 1$. That is, IC_1 is non-binding. ■

Substituting $T_1 = \sqrt{q_1}$ and $T_2 = 4\sqrt{q_2} - 3\sqrt{q_1}$ into objective function, we have:

$$\max_{q} E\pi = \lambda[\sqrt{q_1} - q_1] + (1-\lambda)[4\sqrt{q_2} - 3\sqrt{q_1} - q_2]$$

FOC:

$$\frac{\partial E\pi}{\partial q_1} = \lambda[\frac{1}{2\sqrt{q_1}} - 1] + (1+\lambda)[-\frac{3}{2\sqrt{q_1}}] = 0$$

$\Rightarrow \sqrt{q_1} = \frac{4\lambda-3}{2\lambda}$
if $\sqrt{q_1} > 0 \Rightarrow \lambda > \frac{3}{4} \Rightarrow q_1^* = (\frac{4\lambda-3}{2\lambda})^2 \Rightarrow T_1^* = \frac{4\lambda-3}{2\lambda}$
if $\sqrt{q_1} \leq 0 \Rightarrow \lambda \leq \frac{3}{4} \Rightarrow q_1^* = 0 \Rightarrow T_1^* = 0$

$$\frac{\partial E\pi}{\partial q_2} = \frac{2}{\sqrt{q_2}} - 1 = 0$$

$q_2^* = 4$ and $T_2^* = 8 - 3T_1^*$.

(b) $q_2^* = 4$, $T_2^* = 8$.

(c) If $\lambda > 3/4$, then $(q_1^*, T_1^*) = ((\frac{4\lambda-3}{2\lambda})^2, \frac{4\lambda-3}{2\lambda})$ and $(q_2^*, T_2^*) = (4, 8 - 3\frac{4\lambda-3}{2\lambda})$. If $\lambda \leq 3/4$, then $(q_1^*, T_1^*) = (0,0)$ and $(q_2^*, T_2^*) = (4, 8)$.

(d) If the number of type-θ_1 consumers is large enought($\lambda > 3/4$), monopolist's optimal decision is to discriminate between type-θ_1 and type-θ_2 consumers, otherwise it seizes all type-θ_2 consumers' surplus only.

(e) Monopolist's profit maximization problem:

$$\max_{T,q} \int_{\underline{\theta}}^{\overline{\theta}} [T(q(\theta)) - c(q(\theta))] f(\theta) d\theta$$

subject to:

$$\text{IR: } \theta\sqrt{q(\theta)} - T(q(\theta)) \geq 0, \forall\theta \in \Theta$$

$$\text{IC: } u(\theta) = \theta\sqrt{q(\theta)} - T(q(\theta)) \geq \theta\sqrt{q(\theta')} - T(q(\theta')), \forall\theta' \in \Theta$$

Let θ_0 be the highest type such that the IR constraint is binding. That is , for all $\theta \in [\underline{\theta}, \theta_0]$ we have $\theta\sqrt{q(\theta)} - T(q(\theta)) = 0$. For all $\theta \in [\theta_0, \bar{\theta}]$, consumer's IC constraint requires that

$$u(\theta) = \theta\sqrt{q(\theta)} - T(\theta) = \max_{\alpha\in\Theta} \theta\sqrt{q(\alpha)} - T(q(\alpha))$$

Using the Envelop theorem, we have

$$\frac{du}{d\theta} = \sqrt{q(\theta)}.$$

Integrating above eqution and the fact that $u(\theta) = \theta\sqrt{q(\theta)} - T(q(\theta))$, we have the relationship between $T(q(\theta))$ and $q(\theta)$:

$$T(q(\theta)) = \theta\sqrt{q(\theta)} - \int_{\theta_0}^{\theta} \sqrt{q(t)} dt.$$

Now we can rewrite monopolist's optimal problem as:

$$\max_{q(\theta)} \int_{\theta_0}^{\overline{\theta}} [\theta\sqrt{q(\theta)} - \int_{\theta_0}^{\theta} \sqrt{q(t)} dt - q(\theta)] f(\theta) d\theta.$$

Integrating by parts:

$$\max_{q(\theta)} \int_{\theta_0}^{4} \{\frac{1}{3}[\theta\sqrt{q(\theta)} - q(\theta)] - \sqrt{q(\theta)}(1 - \frac{\theta-1}{3})\} d\theta.$$

$$\max_{q(\theta)} \frac{1}{3} \int_{\theta_0}^{4} [(2\theta - 4)\sqrt{q(\theta)} - q(\theta)] d\theta$$

First, let us look at the optimal q for each θ.

$$\max_{q} (2\theta - 4)\sqrt{q} - q$$

If $2\theta - 4 < 0$, then choosing $q = 0$ is the optimal. Therefore, we know that $\theta_0 = 2$
For $\theta \in [2, 4]$, FOC implies that

$$\frac{1}{2}(2\theta - 4)\frac{1}{\sqrt{q}} = 1$$

$\Rightarrow q(\theta) = (\theta - 2)^2$
$\Rightarrow T(q) = \theta\sqrt{q(\theta)} - \int_2^\theta \sqrt{q(t)}dt = \theta(\theta - 2) - \int_2^\theta (t - 2)dt = \frac{1}{2}\theta^2 - 2 = \frac{1}{2}(\sqrt{q} + 2)^2 - 2 = \frac{1}{2}q + 2\sqrt{q}$
Hence, the monopolist should offer $(q, T(q)) = (q, \frac{1}{2}q + 2\sqrt{q})$ and $q \in [0, 4]$.