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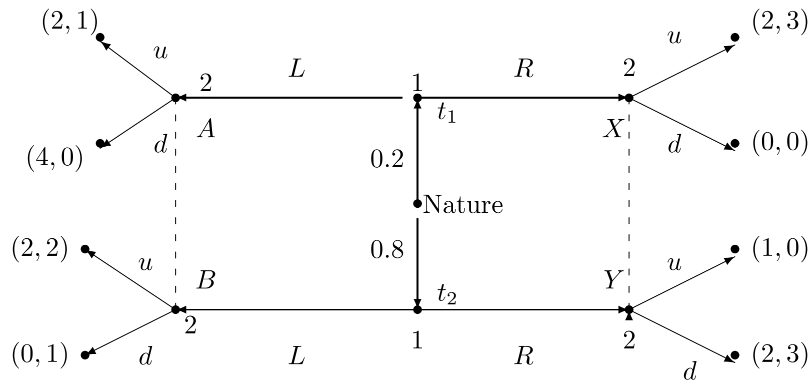
NTU Homework Solutions 02: Equilibrium Refinements

Q: Consider the following extensive form game.



1. Write down each player's decision nodes.

   Ans: Player 1's decision nodes: $\{t_1, t_2\}$. Player 2's decision nodes: $\{A, B, X, Y\}$.

2. Write down each player's information sets.

   Ans: Player 1's information sets: $\{t_1\}$ and $\{t_2\}$. Player 2's information sets: $\{A, B\}$ and $\{X, Y\}$.

3. Write down each player's strategy space.

   Ans: Player 1's strategy space: $\sigma_1(t_1) \times \sigma_1(t_2) = \{LL, LR, RL, RR\}$. Player 2's strategy space: $\sigma_2(\{A, B\}) \times \sigma_2(\{X, Y\}) = \{uu, ud, du, dd\}$.

4. Write down the normal form of this game

   Ans:

		Player 2			
		uu	ud	du	dd
	LL	$2, 1.8$	$2, 1.8$	$0.8, 0.8$	$0.8, 0.8$
Player 1	LR	$1.2, 0.2$	$2, 2.6$	$1.6, 0$	$2.4, 2.4$
	RL	$2, 2.2$	$1.6, 1.6$	$0.4, 1.4$	$0, 0.8$
	RR	$1.2, 0.6$	$1.6, 2.4$	$1.2, 0.6$	$1.6, 2.4$

5. Write down the agent normal form of this game.

   Ans:

		Player 2			
		uu	ud	du	dd
Player 1 at t_1	L	1, 1.8, 2	1, 1.8, 2	4, 0.8, 0	4, 0.8, 0
	R	2, 2.2, 2	0, 1.6, 2	2, 1.4, 0	0, 0.8, 0

L (player 1 at t_2)

		Player 2			
		uu	ud	du	dd
Player 1 at t_1	L	1, 0.2, 1	1, 2.6, 2	4, 0, 1	4, 2.4, 2
	R	2, 0.6, 1	0, 2.4, 2	2, 0.6, 1	0, 2.4, 2

R (player 1 at t_2)

6. Find all the Nash Equilibria of this game.

   Ans:

   - pure strategy Nash equilibrium: $\{(LL, uu), (RL, uu), (LL, ud), (LR, ud)\}$.
   - Observe that the strategy du for player 2 is strictly dominated by the strategy ud. Hence the original game is strategically equivalent to the game in which du is eliminated. Now, we claim that in the remaining game the strategy RR for player 1 is strictly dominated by the mixed strategy $xLL \oplus (1-x)LR$, $\forall x \in (0, 3/4)$. Hence we may again reduce the game by eliminating the strategy RR for player 1. Clearly, in the remaining game the strat-

egy dd for player 2 is strictly dominated by the strategy ud. Finally, we have a 3×2 game.

		Player 2	
		uu	ud
	LL	2,1.8	2,1.8
Player 1	LR	1.2,0.2	2,2.6
	RL	2,2.2	1.6,1.6

Let q_1 denote the probability with player 1 selects LL, q_2 the probability with player 1 selects LR, and $q_3 = 1 - q_1 - q_2$ the probability with player 1 selects RL. Let p_1 denote the probability with player 2 selects uu, and $p_2 = 1 - p_1$ the probability with player 2 selects ud. Then the expected payoff of player 1 equals

pure strategy	expected payoff to player 1
LL	2
LR	$2 - 0.8p$
RL	$0.4p_1 + 1.6$

Therefore LL is a best response for player 1 iff $p_1 \in [0, 1]$, LR is a best reponse iff $p_1 = 0$, and RL is a best response for player 1 iff $p_1 = 1$.

Next, the expected payoff of player 2 equals

uu	$2.2 - 0.4q_1 - 2q_2$
ud	$1.6 + 0.2q_1 + q_2$

Therefore uu is a best response for player 2 iff $q_1 < 1 - 5q_2$, ud is a best response for player 1 iff $q_1 > 1 - 5q_2$.

To determine the Nash equlibria, we consider three case:

Case 1: If $q_1 < 1 - 5q_2$, then the best response for player 2 is uu. In that case, the best reponse for player 1 is a mixed strategy $q_1 LL \oplus (1 - q_1)RL$, $q_2 = 0$. If player 1 selects complete mixed strategy $q_1 LL \oplus (1 - q_1)RL$, which $q_2 = 0$ and $q_1 \in [0, 1)$, it follows that $q_1 < 1 - 5q_2$.

Case 2: If $q_1 > 1 - 5q_2$, then the best response for player 2 is ud. In that case, the best response for player 1 is a mixed strategy $q_1 LL \oplus q_2 LR$, $q_1 + q_2 = 1$. If $q_1 < 1$, then $q_1 < 1 - 5q_2$ is true.

Case 3: If $q_1 = 1 - 5q_2$, then the best reponse for player 2 is a mixed strategy $puu \oplus (1 - p_1)ud$, $p_1 \in (0,1)$. In that case, the best reponse for player 1 is LL, and then it follows that $q_1 = 1 - 5q_2$ is true.

We summarize the discussion above and yield three Nash equilibrium as follows:

$$NE_1 : \begin{cases} q_1 > 0 \quad q_2 = 0 \qquad\qquad q_3 = 1 - q_1 - q_2 > 0 \\ p_1 = 1 \quad p_2 = 1 - p_2 = 0 \end{cases}$$

$$NE_2 : \begin{cases} q_1 > 0 \quad q_2 > 0 \quad q_3 = 0 \\ p_1 = 0 \quad p_2 = 1 \end{cases}$$

$$NE_3 : \begin{cases} q_1 = 1 \quad q_2 = 0 \quad q_3 = 0 \\ p_1 > 0 \quad p_2 > 0 \end{cases}$$

7. Classify the Nash equilibrium outcomes by the following equilibrium concepts:

   (a) outcomes that can be supported by perfect equilibrium of the normal form of this game

   Ans:

   Check the pure Nash equilibrium (LL, uu). Given player 1's strategy is $((1 - \epsilon_1 - \epsilon_2 - \varepsilon_3)LL \oplus \epsilon_1 LR \oplus \epsilon_2 RL \oplus \varepsilon_3 RR)$ and player 2's strategy is $((1 - \epsilon_1' - \epsilon_2' - \varepsilon_3')uu \oplus \epsilon_1' ud \oplus \epsilon_2' du \oplus \varepsilon_3' dd)$. LL is a best response for player 1 for small enough ε, : uu is a best response for player 2 for small enough ε'

   There are three pure Nash equilibrium can be supported by perfect equlibrium of the normal form of this game: $\{(LL, uu), (LL, ud)\}$, (LR, ud)..

   (b) outcomes that can be supported by proper equilibrium of the normal form of this game

   Ans:

   Check the pure Nash eqilibrium (LL, uu). Given player 1's strategy is $((1 - \epsilon - 2\epsilon^2)LL \oplus \epsilon^2 LR \oplus \epsilon RL \oplus \epsilon^2 RR)$ and player 2's strategy is $((1 - \epsilon - 2\epsilon^2)uu \oplus \epsilon ud \oplus \epsilon^2 du \oplus \epsilon^2 dd)$, LL and uu are both best responses for play 1 and player 2, respectively.

   There are three pure strategy Nash equilibria can be supported by proper equlibrium of the normal form of this game: $\{(LL, uu), (LL, ud)\}, (LR, ud)$.

(c) outcomes that can be supported by subgame perfect

Ans: Since there is no proper subgame, all pure Nash equilibrium can be supported by subgame perfect equilibrium.

(d) outcomes that can be supported by sequential equilibrium of this game

Ans:

Case 1: Check the Nash equilibrium (LL, uu).

Take $\sigma_1^\varepsilon(t_1) = (1-5\varepsilon, 5\varepsilon)$ and $\sigma_1^\varepsilon(t_2) = (1-\varepsilon, \varepsilon)$. Hence, player 2's belief μ_2 is given by $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(A) = 0.2$, $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(B) = 0.8$, $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(X) = \frac{5}{9}$ and $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(Y) = \frac{4}{9}$. Given system belief μ_2, u is a best response for player 2 at both $\{A, B\}$ and $\{X, Y\}$. Given player 2's strategy, LL is a best response for player 1.

Case 2: Check the Nash equilibrium (RL, uu).

$\mu_2(A) = 0, \mu_2(B) = 1, \mu_2(X) = 1, \mu_2(Y) = 0$. Given μ_2, u is a best response for player 2 at both $\{A, B\}$ and $\{X, Y\}$. Given player 2's strategy, RL is a best response for player 1.

Case3: Check the Nash equilibrium (LL, ud).

Take $\sigma_1^\varepsilon(t_1) = (1-\varepsilon, \varepsilon)$ and $\sigma_1^\varepsilon(t_2) = (1-\varepsilon, \varepsilon)$. Hence, player 2's belief μ_2 is given by $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(A) = 0.2$, $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(B) = 0.8$, $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(X) = \frac{1}{5}$ and $\lim_{\varepsilon\to 0}\mu_2^\varepsilon(Y) = \frac{4}{5}$. Given system belief μ_2, u and d are best responses for player 2 at $\{A, B\}$ and $\{X, Y\}$, respectively. Given player 2's strategy, LL is a best response for player 1.

Case 4: Check the Nash eqilibrium (LR, ud).

$\mu_2(A) = 1, \mu_2(B) = 0$, $\mu_2(X) = 0, \mu_2(Y) = 1$. Given system belief μ_2, u and d are best responses for player 2 at $\{A, B\}$ and $\{X, Y\}$, respectively. Given player 2's strategy, LR is a best response for player 1.

There are four pure strategy Nash equilibria can be supported by sequential equlibrium of this game: $\{(LL, uu), (RL, uu), (LL, ud), (LR, ud)\}$.

(e) outcomes that can be supported by perfect equilibrium of this game

Ans:

Case 1: Check the Nash equilibrium (LL, uu).

Let player 2's strategy be $((1-\epsilon_1)u \oplus \epsilon_1 d)$ at $\{A, B\}$ and $((1-\epsilon_2)u \oplus \epsilon_2 d)$ at $\{X, Y\}$. LL is a best response for player 1 if $\varepsilon_2+2\varepsilon_1 \leq 1$. Given player 1's strategy is $((1-\epsilon_3)L \oplus \epsilon_3 R)$ at t_1 and $((1-\epsilon_4)L \oplus \epsilon_4 R)$ at t_2. Player 2's belief at $\{X,Y\}$ is $\mu_2(X) = \frac{\epsilon_3}{\epsilon_3+4\epsilon_4}$ and $\mu_2(Y) = \frac{4\epsilon_4}{\epsilon_3+4\epsilon_4}$. u is a best response for player 2 at $\{X,Y\}$ iff $\varepsilon_3 \geq 4\varepsilon_4$..

Case 2: Check the Nash equilibrium (RL, uu).

Let player 2's strategy be $((1-\epsilon_1)u \oplus \epsilon_1 d)$ at $\{A,B\}$ and $((1-\epsilon_2)u \oplus \epsilon_2 d)$ at $\{X,Y\}$. R is a best response for player 1 at t_1 iff $\varepsilon_1+\varepsilon_2 < 0$, which is not possible.

Case3: Check the Nash equilibrium (LL, ud).

Let player 2's strategy be $((1-\epsilon_1)u \oplus \epsilon_1 d)$ at $\{A,B\}$ and $((1-\epsilon_2)d \oplus \epsilon_2 u)$ at $\{X,Y\}$. LL is a best response for player 1 iff $2\varepsilon_1 \leq \varepsilon_2$. Given player 1's strategy is $((1-\epsilon_3)L \oplus \epsilon_3 R)$ at t_1 and $((1-\epsilon_4)L \oplus \epsilon_4 R)$ at t_2. Player 2's belief at $\{X,Y\}$ is $\mu_2(X) = \frac{\epsilon_3}{\epsilon_3+4\epsilon_4}$ and $\mu_2(Y) = \frac{4\epsilon_4}{\epsilon_3+4\epsilon_4}$. d is a best response for player 2 at $\{X,Y\}$ iff $\varepsilon_3 \leq 4\varepsilon_4$..

Case 4: Check the Nash eqilibrium (LR, ud).

Given player 2's strategy is $((1-\epsilon_1)u \oplus \epsilon_1 d)$ at $\{A,B\}$ and $(\epsilon_2 u \oplus (1-\epsilon_2)d)$ at $\{X,Y\}$. L is a best response for player 1 at t_1. R is a best response for player 1 at t_2 iff $\epsilon_2 - 2\epsilon_1 < 0$. Given player 1's strategy is $((1-\epsilon_3)L \oplus \epsilon_3 R)$ at t_1 and $(\epsilon_4 L \oplus (1-\epsilon_4)R)$ at t_2. Player 2's belief at $\{X,Y\}$ is $\mu_2(X) = \frac{\epsilon_3}{\epsilon_3+4(1-\epsilon_4)}$ and $\mu_2(Y) = \frac{4(1-\epsilon_4)}{\epsilon_3+4(1-\epsilon_4)}$. d is a best response for player 2 at $\{X,Y\}$ iff $\varepsilon_3+4\varepsilon_4 \leq 4$.

There are four pure Nash equilibria can be supported by perfect equlibrium of this game: $\{(LL, uu), (LL, ud), (LR, ud)\}$.