A Simple Closed-Form Approximation for the Cumulative Distribution Function of the Composite Error of Stochastic Frontier Models

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Abstract

This paper derives an analytic closed-form formula for the cumulative distribution function (cdf) of the composite error of the stochastic frontier analysis (SFA) model. Since the presence of a cdf is frequently encountered in the likelihood-based analysis with limited-dependent and qualitative variables as elegantly shown in the classic book of Maddala (1983), the proposed methodology is useful in the framework of the stochastic frontier analysis. We apply the formula to the maximum likelihood estimation of the SFA models with a censored dependent variable. The simulations show that the finite sample performance of the maximum likelihood estimator of the censored SFA model is very promising. A simple empirical example on the modeling of reservation wage in Taiwan is illustrated as a potential application of the censored SFA.

Key words: Stochastic frontier analysis, cumulative distribution function, censored stochastic frontier model

1. Introduction

This paper derives an analytic closed-form formula for the cumulative distribution function (cdf) of the composite error in the stochastic frontier analysis (SFA) introduced by Aigner et al. (1977). The SFA regressions take the form, $y = f(x;\beta) + v + u$, where the two-sided random error v represents statistical noise and $u \geq 0$ reflects managerial inefficiency. The composite error $\varepsilon = v + u$ is the major feature of the SFA model and plays an important role on the associated analysis. The standard maximum likelihood estimation of the SFA requires the evaluation of the probability density function (pdf) of the composite error ε. The computation of the pdf of ε is relatively easy and routine. Nevertheless, there is no simple way to approximate the cumulative distribution function of the composite error, even though the cdf is frequently encountered in the studies with limited-dependent and qualitative variables as clearly demonstrated in Maddala (1983). Many studies, for instance, Hofler and Murphy (1992, 1994), and Polachek and Robst (1998), have applied the SFA framework to estimate reservation wages of workers and the effects of labor market imperfection information on wages. Had the wage regression $y = f(x;\beta) + \varepsilon$ were censored with the minimum wage restriction, the model would correspond to either the censored or the truncated SFA regression and the maximum likelihood estimation would then require the computation of the cdf of the composite error. Recently, Park and Lohr (2007) propose a deterministic frontier model to evaluate the performance extension service providers in the US land grant universities using order response data on the dependent variable y. Had the model were SFA, the maximum likelihood estimation of the binary or ordinal stochastic frontier regression would again require the computation of the cdf of the composite error. The current paper fills this gap in the literature by proposing an analytic closed-form formula for the cdf of the SFA composite error.

Apparently, the applicability of the proposed approximation method is far reaching. Recently, Amsler et al. (2011) consider the stochastic frontier models in the setting where there is correlated inefficiency over time; and Lai and Huang (2011) consider the multiple stochastic frontier models with correlated composite errors in the setting of seemingly unrelated regressions. In these settings, they recognize that current methods of modeling dependence are either restrictive or computationally infeasible. They thus suggest using a copula to the composite errors of the SFA model in order to replace the task of multi-dimensional integration of the traditional methods with a much easier one-dimensional integral. Interestingly, this one-dimensional integral happens to be the cdf of the composite error which serves as an argument of the copula function. Indeed, Amsler et al. (2011) and Lai and Huang (2011) recognize that the approximation method proposed in this paper can speed up the evaluations of their copula-based approach.

The remaining parts of this paper are arranged as follows: In section 2 we present a closed-form formula for computing the cdf of the composite error. The accuracy of the proposed formula is examined via an empirical distribution of ten million random drawings of the composite error. In section 3, we apply the formula to derive the likelihood function of the SFA with a censored dependent variable to illustrate the power of the proposed formula. Section 4 provides a simple application of the proposed closed-form computation of the cdf of the composite error in the estimate of the reservation wages of employed workers in Taiwan where the legislated minimum wage severed as the censored point. Section 5 gives a summary and conclusions.

2. Analytic Close-Form Formula for the CDF of a Composite Error of SFA

Consider a standard linear stochastic frontier model,

$$y_i = x_i^{\mathrm{T}} \beta + \varepsilon_i, \qquad i = 1, 2, \ldots, n \tag{1}$$

where y_i and ε_i are the i^{th} observation on the dependent variable and the random error, respectively; x_i^{T} is a $1\times k$ vector of the i^{th} observation on the k regressors; and β is a $k\times 1$ vector of unknown parameters to be estimated. The composite error ε_i is specified as:

$$\varepsilon_i = v_i + u_i, \tag{2}$$

where the random errors v_i are independently and identically distributed (iid) as $N(0,\sigma_v^2)$, and the random errors u_i are the absolute values of the variables that are iid as $N^{+}(0,\sigma_u^2)$. All v_i's and u_i's are independent of each other, and are also independent of x_i. We follow the reparameterization of Aigner et al. (1977) in setting

$$\sigma^2 = \sigma_u^2 + \sigma_v^2 \text{, and } \lambda = \frac{\sigma_u}{\sigma_v}. \tag{3}$$

The log likelihood function of the model defined in (1)-(3) is shown to be

$$L_0 = \frac{n}{2}\ln\left(\frac{2}{\pi}\right) - n\ln(\sigma) + \sum_{i=1}^{n}\ln\left[\Phi\left(\frac{\lambda}{\sigma}\varepsilon_i\right)\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\varepsilon_i^2, \tag{4}$$

where $\varepsilon_i = y_i - x_i^{\mathrm{T}}\beta$, and Φ is the cdf of $N(0,1)$. The maximum likelihood estimation is obtained by the maximization of (4) with respect to the parameter $(\beta^{\mathrm{T}}, \lambda, \sigma)$.

As clearly demonstrated by Maddala (1983), the cdf is frequently encountered in the context of limited-dependent and qualitative variables, including the censored regression model, the self-selectivity model of Roy (1951), and the SFA with sample selection of Greene (2010). Without an accurate cdf, the resulting analysis will be incorrect, or even impossible.

To clarify the basic idea of this paper, we employ the case of censored regression as the workhorse. The standard censored (Tobit) model (Tobin, 1958), i.e., the dependent variable is censored, has been wildly employed in the literature. The Tobit model takes the general form, $y = f(X;\beta) + v$, where the noise v is usually assumed to be normally distributed. Under this set-up, the maximum likelihood estimation of the Tobit model can be easily implemented with many computer packages. When the dependent variable y in the SFA regression contains a considerable number of censored observations, then the stochastic frontier regression is of the censored type rather than the standard (uncensored) SFA model.

Without loss of generality, we assume the point of censoring is at 0 throughout this paper, i.e.,

$$\begin{cases} y_i^* = x_i^{\mathrm{T}}\beta + \varepsilon_i, & i=1, 2, ..., n; \\ y_i = y_i^*, & \text{if } \mathrm{y}_i^* > 0; \\ y_i = 0, & \text{if } \mathrm{y}_i^* \le 0. \end{cases} \tag{5}$$

where $\varepsilon_i = v_i + u_i$. It is inappropriate to estimate the parameters of model (5) via the log likelihood function L_0 in (4) because of the presence of the censored dependent variable. Thus, L_0 is called the standard (uncensored) SFA log likelihood function. As in

Amemiya (1985), the censored SFA log likelihood function for *n* independent observations of model (5) is:

$$L_1 = \sum_{y_i>0} \ln\left(f\left(\varepsilon_i\right)\right) + \sum_{y_i=0} \ln\left(F\left(-x_i^{\mathrm{T}}\beta\right)\right), \tag{6}$$

where $f(.)$ and $F(.)$ are the density and distribution function of the composite error $\varepsilon_i = v_i + u_i$, respectively. The first summation is over the observations for which $y_i > 0$ and the second summation is over the observations for which $y_i = 0$.

From the estimation point of view, the uncensored part in (6) is easy to compute because the density $f(\varepsilon_i)$ is well-known,

$$f\left(\varepsilon_i\right) = \frac{2}{\sigma}\phi\left(\frac{\varepsilon_i}{\sigma}\right)\Phi\left(\frac{\lambda}{\sigma}\varepsilon_i\right), \tag{7}$$

where $\phi(.)$ denotes the density function of *N*(0,1). However, the difficulty in the maximization of (6) is in computing the censored part,

$$F\left(-x_i^{\mathrm{T}}\beta\right) = \int_{-\infty}^{-x_i^{\mathrm{T}}\beta} f\left(\varepsilon\right)d\varepsilon, \qquad \text{for } y_i = 0. \tag{8}$$

Nevertheless, there exists no analytic method to approximate the cdf in (8).

Note that the above distribution function $F(.)$ can be expressed as:

$$F\left(Q\right) = \frac{2}{\sigma} I\left(Q\right), \tag{9}$$

where

$$I\left(Q\right) = \int_{-\infty}^{Q}\left(\int_{-\infty}^{a\varepsilon}\phi\left(\varsigma\right)d\varsigma\right)\phi\left(b\varepsilon\right)d\varepsilon \tag{10}$$

and $a = \dfrac{\lambda}{\sigma}$, $b = \dfrac{1}{\sigma}$, $Q = -x_i^{\mathrm{T}}\beta$. In this paper, we derive an approximated formula $I_{app}\left(Q\right)$ for $I\left(Q\right)$ in (10). However, we note here the error function $erf(z)$ is defined as:

$$erf\left(z\right) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt = 2\int_0^{\sqrt{2}z}\phi\left(t\right)dt. \tag{11}$$

Under (Q, a, b) $\in$ finite R, $a \geq 0$, and $b > 0$, $I(Q)$ in (10) can be approximated by $I_{app}(Q)$:

$$I_{app}(Q) = exp\left(\frac{a^2 c_1^2}{4b^2 - 4a^2 c_2}\right) \frac{1}{4\sqrt{b^2 - a^2 c_2}} \left[1 - erf\left(\frac{-a\ c_1 + \sqrt{2} Q\left(b^2 - a^2 c_2\right) sign(Q)}{2\sqrt{b^2 - a^2 c_2}}\right)\right]$$

$$+ \frac{erf\left(\frac{b\ Q}{\sqrt{2}}\right)}{2b} \frac{1 + sign(Q)}{2}, \tag{12}$$

where $c_1 = -1.09500814703333$ and $c_2 = -0.75651138383854$. The derivation of I_{app} is given in the appendix. Given I_{app}, the cdf $F(Q)$ in (9) can then be approximated by:

$$F_{app}(Q) = \frac{2}{\sigma} I_{app}(Q). \tag{13}$$

Basically, the proposed formula I_{app} involves the error function $erf(z)$, which can easily be computed with the standard statistical package. Accordingly, the computation of F_{app} is extremely straightforward.

Mathematically, the role of the two constants, c_1 and c_2 is to ensure that the error function $erf(z)$ can be well approximated by another function, $g(z) = 1 - e^{c_1 z + c_2 z^2}$, for $z \geq 0$. The choice of c_1 and c_2 is to make the two functions, $erf(z)$ and $g(z)$, as close to each other as possible. One possible method is to use the first-order Taylor expansion around z = 1 to obtain the values of c_1 and c_2. An alternative approach, as adopted in this paper, is to use the nonlinear least squares method to estimate c_1 and c_2 based on 500 equally spaced points within the interval $[0, 5]$. The interval is chosen because $erf(0) = 0$ is the lower limit of the error function, and $erf(5) \approx 1 - 0.15 \times 10^{-8}$ is very close to the upper limit, $erf(\infty) = 1$. The estimated c_1 and c_2 are both negative. This choice of interval and function $g(z)$ implies that $g(0) = 0$, and $g(5) \approx 1 - 0.3 \times 10^{-10}$, which is approximately 1. Moreover, both $g(z)$ and $erf(z)$ are monotonically increasing functions. Thus, the error function $erf(z)$ is well approximated by $g(z)$ for z

≥ 0. Indeed, we find the maximum difference between $erf(z)$ and $g(z)$ is 0.002205 for the argument being 0,0.0001,…,3.

Table 1 demonstrates that $F_{app}(Q) = \frac{2}{\sigma} I_{app}(Q)$ delivers a very accurate approximation to $F(Q)$ which cannot be exactly known, but can be estimated by the Accept-Reject algorithm based on a large number of independent draws of ε. Ten million random drawings of ε are observed and the cumulative distribution $F(Q)$ is estimated from the empirical distribution of $\varepsilon \leq Q$. Indeed, for various choices of Q and parameter sets of σ_u and σ_v, the absolute difference between $F_{app}(Q)$ and the empirical estimate of $F(Q)$ based on the Accept-Reject algorithm is less than 0.0003 in probability. More importantly, the absolute difference $\left|F_{app}(Q) - F(Q)\right|$ exhibits no apparent pattern either at the truncation point Q, or the values of parameters σ_u and σ_v. The good approximation of $F_{app}(Q)$ to $F(Q)$ also explains the excellent finite sample performance of the MLE of the censored SFA under various model configurations considered in the next section.

[insert Table 1 here]

3. Maximum Likelihood Estimation of Censored Stochastic Frontier Regressions

In this section we consider the finite sample performance of the MLE based on the standard SFA likelihood function, L_0, and the MLE based on the censored SFA likelihood function, L_1, when the data-generating processes (DGP) are model (5). The focal point is on the effects of the formula in (12) on the likelihood-based estimation.

Following Olson et al. (1980), we consider a set of experiments with a simple model:

$$y_i^{*l} = \beta_0 + \beta_1 x_i + \varepsilon_i^l, \quad i = 1, 2, \ldots, n, \; l = 1, 2, \ldots, 500, \tag{14}$$

where l denotes the l-th replication of the data, and the regressors x are drawn from $N(0,1)$. The parameters considered in the Monte Carlo experiments are[1]:

$$\xi = \left(\beta_0, \beta_1, \sigma_u, \sigma_v\right)^{\mathrm{T}}.$$

The maximization of the standard SFA likelihood function L_0 in (4) is well established. For the censored SFA likelihood function L_1 in (6), we take the derivatives of the approximated formula with respect to the parameter ξ, i.e.,

$$\frac{\partial L_1}{\partial \xi} \approx \sum_{y_i>0} \frac{\partial \ln\left(f\left(\varepsilon_i\right)\right)}{\partial \xi} + \sum_{y_i=0} \frac{\partial \ln\left(F_{app}\left(-x_i^{\mathrm{T}}\beta\right)\right)}{\partial \xi}. \tag{15}$$

All the programs are written in GAUSS. The optimization algorithm used to implement the MLE is the quasi-Newton algorithm of Broyden, Fletcher, Goldfarb, and Shanno (BFGS) contained in the GAUSS MAXLIK library[2]. The maximum number of iterations for each replication is 200. In order to create a more realistic scenario in simulation, the initial value for the MLE procedure is set at the true parameter value plus a random number generated from $N(0,1)$[3]. Furthermore, to provide a fair comparison between L_0 and L_1 estimators, we record the first 500 replications with normal convergence for numerical analysis.

The experimental design intends to show that a significant bias exists in MLE if the presence of censored dependent variable is not taken into account. Intuitively, the more observations are censored, the more weight the censored part of the likelihood function L_1 in (6) carries in the maximization. Consequently, a larger bias in MLE is expected based on the misspecified standard SFA model. In particular, consider the probability limit of the derivative of L_0 with respect to ξ,

$$Plim\ \frac{1}{n}\frac{\partial L_0}{\partial \xi} = \left(1-m\right) Plim\ \frac{1}{n_1}\sum_{y_i>0}\frac{\partial \ln f\left(\varepsilon_i\right)}{\partial \xi} + m\ Plim\ \frac{1}{n_0}\sum_{y_i=0}\frac{\partial \ln f\left(\varepsilon_i\right)}{\partial \xi}$$

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1 In the experiments, we estimate ξ with the following transformation function: $\xi = \left(\beta_0, \beta_1, \sigma_u, \sigma_v\right)^{\mathrm{T}} = \kappa\left(\tilde{\xi}\right)$, where $\tilde{\xi} = \left(\beta_0, \beta_1, \ln\left(\sigma_u\right), \ln\left(\sigma_v\right)\right)^{\mathrm{T}}$ are the parameters actually estimated when conducting the MLE of the censored and uncensored SFA.

2 The GAUSS program for the censored SFA is available upon request from the authors.

3 More precisely, the initial value of $\tilde{\xi}$ is set to be $\tilde{\xi}_0 = \kappa(\tilde{\xi})^{-1} + N(0,1)$.

where *m* is the probability and n_0 is the observations that, $y_i = 0$ respectively, and $n_1 + n_0 = n$. If the true model is of the censored SFA, we have $Plim\ \frac{1}{n_1}\sum_{y_i>0}\frac{\partial \ln f(\varepsilon_i)}{\partial \xi} = 0$, but $Plim\ \frac{1}{n_0}\sum_{y_i=0}\frac{\partial \ln f(\varepsilon_i)}{\partial \xi} \neq 0$. Thus, the degree of bias in the standard SFA depends on the probability *m* that $y_i = 0$.

To examine the sensitivity of the MLE with respect to censored observations, various percentages of the latent dependent variable (y_i^*) falling below zero are assumed in the Monte Carlo experiments. To this end, the slope of the censored SFA in (14) is assumed to be either zero, $\beta_1 = 0$, or one, $\beta_1 = 1$, for all experiments, and the intercept β_0 is set to an appropriate value to ensure the probability that y_i^* falling below zero is at a specified m. More specifically, in an experiment with a pre-assigned m, the intercept can be set as $\beta_0 = -F^{-1}(m)$, where $F^{-1}(.)$ is the inverse distribution function of ε , and the parameters considered for the simulations are:

$$\xi = (\beta_0, \beta_1, \sigma_u, \sigma_v)^{\mathrm{T}} = (-F^{-1}(m), 0, \sigma_u, \sigma_v)^{\mathrm{T}}. \tag{16}$$

All Monte Carlo experiments are conducted with 500 replications. Some easily distinguished patterns of bias in the MLE emerge in Tables 2-3. The results clearly show that in almost all cases, the bias resulting from the standard MLE (L_0) of the parameters is much larger than the bias resulting from the censored MLE (L_1), and as expected, the bias from L_0 seems to increase with the degree of censoring *m* as we can clearly see from the changing pattern of the estimated σ_u and σ_v. Furthermore, the bias from the standard MLE (L_0) of the standard SFA is considerable relatively to its true value, especially in β_0, σ_u, and σ_v. On the other hand, it appears that the finite sample performance of the censored MLE based on the proposed approximated formula F_{app} in (13) is very promising, as the associated bias is negligible relative to its true value. This outcome is direct evidence of the effectiveness of applying F_{app} in the maximum likelihood estimation of censored stochastic frontier models.

From our experience in conducting Monte Carlo experiments, the MLE of the standard SFA regression often fails to obtain “normal” convergence in computation when the probability of censoring or the number of censoring observations is, for example, larger than 8%. The failure is due to either the number of iterations in maximization beyond a reasonable limit or, in most cases, the iterative estimates of σ_u (or σ_v) tending to approach to the boundary of the parameter space, i.e., 0. On the other hand, however, the failure of normal convergence seldom occurs in the maximization of L_1 when the sample size is greater than 100.

[insert Tables 2-3 here]

In Tables 4 and 5 we illustrate the corresponding mean squared errors (MSE) from the design in Table 2 and Table 3. The MSE of censored SFA always decreases with increasing sample size, revealing the censored MLE based on our proposed analytic formula in (13) possesses a well-defined asymptotic behavior. On the other hand, we cannot see such a pattern from the standard SFA. In particular, when $\beta_0 = 0.53\ (m = 0.06)$, MSE is 0.0781 as the sample size is 400, and the value changes to be 0.0824 when the sample size increases to be 800.

[insert Tables 4-5 here]

To further demonstrate the effectiveness of applying F_{app} in the MLE of the censored SFA estimation, we conduct more detailed experiments with various combinations of parameters $\xi = \left(\beta_0, \beta_1, \sigma_u, \sigma_v\right)^{\mathrm{T}}$, and sample size. The results in Tables 6-8 show the MSE of the MLE of ξ. For all 27 combinations of ξ considered in Tables 6-8, the MSE of censored SFA always decreases with increasing sample size. The results again assure the promising performance of the proposed F_{app} approximation formula. This finding is important, but not surprising in that the MLE of censored SFA takes into account the presence of censored dependent variable in estimation. The simulations enhance our understanding and confidence in the use of F_{app} in the MLE of censored SFA.

[insert Tables 6-8 here]

4. An Empirical Example

According to search theory, workers form "reservation" wages such that the job offer paying a wage higher than the reservation wage is accepted and the job search is terminated. Hofler and Murphy (1992, 1994) model the job search and wage determination process in a stochastic frontier framework,

$$w_i = w_i^r + u_i = x_i^T \beta + v_i + u_i \tag{17}$$

where x_i's are the reservation wage (w_i^r) determinants with coefficients β. The two-sided random error v_i represents statistical noise and the non-negative $u_i \geq 0$ reflects the degree by which the worker's observed wage exceeds the reservation wage. With data on employed workers, the estimation of the reservation wage and the wage premium (the spread between observed and reservation wages) is a straightforward application of the standard stochastic frontier analysis. Econometrically, however, the existence of the regulation of minimum wage complicates the above analysis in that

$$\begin{cases} w_i^* = x_i^T \beta + v_i + u_i \\ w_i = w_i^*, & if\ w_i^* > w^{\min} \\ w_i = w^{\min}, & if\ w_i^* \leq w^{\min} \end{cases} \tag{18}$$

where $w^{\min}$ denotes the minimum wage and serves as the censoring points for the observed wage. Thus, the reservation wage regression with censored minimum wage restriction corresponds to the censored SFA regression. The estimation of reservation wage might then be biased if we do not take the presence of minimum wage regulation into account. The proposed method of this paper is capable of dealing this issue.

The data used in the empirical example reported below were drawn from the Human Resource Survey conducted in 2006 by the government statistical office of Taiwan. The survey employed a stratified two- stage random sampling framework on the general population with ages over 15 excluding those in army service or in prison. To exemplify significance of minimum wage in the modeling of reservation wage, the survey is further screened to include only the full-time workers, excluding self-employed or family workers, in the eastern region of Taiwan. The above screening of the survey data yields a total of 1128 sample workers. The wage variable (w) is the regular 42-week-hours monthly wage earning excluding overtime payment. The minimum wage ($w^{\min}$) in 2006 was NT$15,840, set by the government of Taiwan. The sample shows a total of 194 or 17% of workers working at the minimum wage level.

The dependent variable is the natural logarithm of the worker's monthly wage. The explanatory variables (x) used in estimating the reservation wage are a set of socio-demographic factors. More specifically, the explanatory variables are:

Gov =1 if work for government and Gov = 0, otherwise. Government employees on average are better paid than the private counterpart.

Gender = 1 if female and Gender = 0, otherwise. Female is usually paid less due to sex discrimination in job market.

Marry = 1 if married or cohabitation and Marry =0, otherwise.

Edu = the educational attainment measured in year of schooling divided by 10. Higher education attainment would generate better wage.

Age = worker's age divided by 40 as a measure of work experience. The work experience is postulated to have a positive impact on the reservation wage at a decreasing rate. Therefore, the squared (Age) term is included as an additional explanatory variable.

Farming = 1 if works in farming industry and Farming = 0, otherwise. Workers in farming are in general paid less than those in other industries.

[insert Table 9 here]

Table 9 shows, in the logarithmic form, both estimated reservation wage equations based on the proposed censored SFA specification and the standard (uncensored) SFA specification without the consideration of the minimum wage restriction. In general, the overall fitting for both censored SFA and standard SFA models are reasonably well that all coefficient estimates are statistically significant at the 1% level. However, the censored coefficient estimates, except the coefficient of Marry, are larger in absolute value than the standard SFA estimates. Interestingly, this leads to the empirical finding that the censored SFA predicts higher (lower) reservation wage than the prediction from the standard SFA model when the predicted reservation wage is over (under) the minimum wage of $15,840 NT dollars as shown in Figure 1. Furthermore, by the simulation results in Table 2 and Table 3, under the presence of censored observations, we find a upward bias in the estimate $\hat{\sigma}_u$, and a downward bias in $\hat{\sigma}_v$ if we employ the standard SFA specification. The results in Table 9 reflect these observations in that the estimate $\hat{\sigma}_u$ is larger from the standard SFA than that from the censored SFA, and the converse is true for $\hat{\sigma}_v$.

5. Summary and Conclusions

We propose an easy-to-implement and accurate closed-form formula for computing the cumulative distribution function of the composite error of the SFA model. The Monte Carlo experiments show that the proposed method is powerful to deal with the maximum likelihood estimation of the censored SFA models. Moreover, the failure of normal convergence almost never occurs in the maximization of the correctly specified censored SFA likelihood function based on the proposed formula, indicating the computational stability of our method. Since the presence of the cdf is widely spread in the studies of limited-dependent and qualitative variables, we contemplate that our formula is a useful instrument for the stochastic frontier analysis with these data. A simple empirical example on modeling of reservation wage determination with minimum wage restriction is illustrated. The reservation wage model corresponds to the proposed censored SFA model and, as expected, the empirical findings using a Taiwanese survey data are consistent to the theoretical prediction on the bias of coefficient estimates, particularly in the variance estimates of the error terms.

Appendix

This appendix shows the derivation of the approximated cumulative distribution function F_{app}. Since $a = \dfrac{\lambda}{\sigma} \geq 0$, we divide the derivation into two parts: for $(Q \leq 0, a \geq 0)$ and $(Q \geq 0, a \geq 0)$. Furthermore, for ease of exposition, two equations from Abramowitz and Stegun (1970, equations (7.11) and (7.4.32)) are given:

$$erf\,(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt = 2\int_0^{\sqrt{2}z}\phi(t)\,dt\,,$$

$$\int e^{-\left(kx^2+2mx+n\right)}dx = \frac{1}{2}\sqrt{\frac{\pi}{k}}\;e^{\frac{m^2-kn}{k}}\;erf\left(\sqrt{k}\,x + \frac{m}{\sqrt{k}}\right) + C,\;\; k \neq 0,$$

where C denotes a finite constant.

Given that $(Q, a, b) \in$ finite R, $b > 0$, $erf\,(-x) = -erf\,(x)$, and define $\varepsilon = \sqrt{2}v/a$, we have:

$$\begin{aligned}
I_{a\geq 0} &= \frac{\sqrt{2}}{a}\int_{-\infty}^{\frac{a}{\sqrt{2}}Q}\left(\int_{-\infty}^{\sqrt{2}v}\phi(\varsigma)\,d\varsigma\right)\phi\left(\sqrt{2}v\frac{b}{a}\right)dv \\
&= \frac{\sqrt{2}}{2a}\int_{-\infty}^{\frac{a}{\sqrt{2}}Q}\left(1 + erf\,(v)\right)\,\phi\left(\sqrt{2}v\frac{b}{a}\right)dv \\
&= \frac{\sqrt{2}}{2a}\int_{-\infty}^{0}\left(1 + erf\,(v)\right)\,\phi\left(\sqrt{2}v\frac{b}{a}\right)dv + \frac{\sqrt{2}}{2a}\int_{0}^{\frac{a}{\sqrt{2}}Q}\left(1 + erf\,(v)\right)\,\phi\left(\sqrt{2}v\frac{b}{a}\right)dv,
\end{aligned}$$

Note that $erf\,(z)$ can be well approximated by a function, $g(x) = 1 - e^{c_1x + c_2x^2}$ for $x \geq 0$, where c_1 and c_2 are chosen to ensure that $g(x)$ is as close to $erf\,(x)$ as possible. The choice of c_1 and c_2 is discussed in Section 2.

With the preceding results of $I_{a\geq 0}(Q)$, we then have

$$\begin{aligned}
I_{a\geq 0, Q\geq 0}(Q) &= \frac{\sqrt{2}}{2a}\int_{-\infty}^{0}\left(1 + erf\,(v)\right)\,\phi\left(\sqrt{2}v\frac{b}{a}\right)dv + \frac{\sqrt{2}}{2a}\int_{0}^{\frac{a}{\sqrt{2}}Q}\left(1 + erf\,(v)\right)\,\phi\left(\sqrt{2}v\frac{b}{a}\right)dv \\
&= \frac{\sqrt{2}}{2a}\int_{0}^{\infty}\left(1 - erf\,(v)\right)\,\phi\left(\sqrt{2}v\frac{b}{a}\right)dv + \frac{\sqrt{2}}{2a}\int_{0}^{\frac{a}{\sqrt{2}}Q}\left(1 + erf\,(v)\right)\,\phi\left(\sqrt{2}v\frac{b}{a}\right)dv
\end{aligned}$$

$$\approx \frac{\sqrt{2}}{2a}\int_0^{\infty} e^{c_1 v + c_2 v^2}\ \frac{1}{\sqrt{2\pi}}\ e^{-\frac{v^2 b^2}{a^2}}\,dv + \frac{\sqrt{2}}{a}\int_0^{\frac{a}{\sqrt{2}}Q}\ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{v^2 b^2}{a^2}}\,dv$$

$$-\frac{\sqrt{2}}{2a}\int_0^{\frac{a}{\sqrt{2}}Q}\ \left(e^{c_1 v + c_2 v^2}\right)\frac{1}{\sqrt{2\pi}}\, e^{-\frac{v^2 b^2}{a^2}}\,dv$$

$$= \frac{1}{2\sqrt{\pi}a}\int_0^{\infty} exp\left(-\left(\tfrac{b^2 - a^2 c_2}{a^2}\right)v^2 + c_1 v\right)dv$$

$$+\frac{1}{\sqrt{\pi}a}\int_0^{\frac{b}{\sqrt{2}}Q}\ exp\left(-v^2\right)d\left(\tfrac{a}{b}v\right) - \frac{1}{2\sqrt{\pi}a}\int_0^{\frac{a}{\sqrt{2}}Q} exp\left(-\left(\tfrac{b^2 - a^2 c_2}{a^2}\right)v^2 + c_1 v\right)dv.$$

When we use (7.4.32) of Abramowitz and Stegun (1970), $I_{a\geq 0, Q\geq 0}$ can be approximated by:

$$I_{a\geq 0, Q\geq 0}(Q) \approx exp\left(\frac{a^2 c_1^2}{4b^2 - 4a^2 c_2}\right)\ \frac{1}{4\sqrt{b^2 - a^2 c_2}}\ \left[1 - erf\left(\frac{-a\ c_1 + \sqrt{2}Q\left(b^2 - a^2 c_2\right)}{2\sqrt{b^2 - a^2 c_2}}\right)\right]$$

$$+\frac{1}{2b}erf\left(\frac{b\ Q}{\sqrt{2}}\right).$$

Likewise, we can derive the approximation for $I_{a\geq 0, Q\leq 0}$:

$$I_{a\geq 0, Q\leq 0}(Q) \approx exp\left(\frac{a^2 c_1^2}{4b^2 - 4a^2 c_2}\right)\ \frac{1}{4\sqrt{b^2 - a^2 c_2}}\ \left[1 - erf\left(\frac{-a\ c_1 - \sqrt{2}Q\left(b^2 - a^2 c_2\right)}{2\sqrt{b^2 - a^2 c_2}}\right)\right].$$

Combining the preceding results, we prove the result in (12).

Table 1. Accuracy of $F_{app}(Q)$ in Computing $F(Q)$ at Various Q, σ_u, and λ

$\sigma_u = 1$				Q			
Method	–3.0	–2.0	–1.0	0.0	1.0	2.0	3.0
				$\lambda = 1.50\ (\sigma_v = 0.6667)$			
$100 \times F_{app}(Q)$	0.000060	0.022664	1.488730	18.740960	60.943449	90.408314	98.743208
$100 \times F(Q)$	0.000060	0.019500	1.477160	18.725950	60.914390	90.406290	98.746360
$100 \times (AbsD)$	0.000000	0.003164	0.011570	0.015010	0.029059	0.002024	0.003152
				$\lambda = 1.25\ (\sigma_v = 0.8000)$			
$100 \times F_{app}(Q)$	0.001620	0.128156	2.957624	21.493701	59.469658	88.293175	98.086663
$100 \times F(Q)$	0.001390	0.122430	2.954490	21.481860	59.462650	88.285310	98.088210
$100 \times (AbsD)$	0.000230	0.005726	0.003134	0.011841	0.007008	0.007865	0.001547
				$\lambda = 0.85\ (\sigma_v = 1.1765)$			
$100 \times F_{app}(Q)$	0.149700	1.499332	8.425067	27.601621	56.692885	81.965878	94.941287
$100 \times F(Q)$	0.147350	1.499210	8.428980	27.579250	56.708140	81.971100	94.934540
$100 \times (AbsD)$	0.002350	0.000122	0.003913	0.022371	0.015255	0.005222	0.006747

Note: $F_{app}(Q)$ is computed based on $\frac{2}{\sigma} I_{app}(Q)$ in (13) and $F(Q)$ is computed from the Accept-Reject algorithm based on 10 million independent draws of the distribution $f(\varepsilon)$ in (7). *AbsD* denotes the absolute difference $\left|F_{app}(Q) - F(Q)\right|$.

Table 2. Bias of the Censored (L_1) and Standard (L_0) MLE:

$\beta_0 = -F^{-1}(m),\ \beta_1 = 0,\ \sigma_v = 0.6667,\ \sigma_u = 1$

	β_0		β_1		σ_u		σ_v	
L	L_0	L_1	L_0	L_1	L_0	L_1	L_0	L_1
n				$\beta_0 = 0.91\ (m = 0.02)$				
100	0.0139	0.1240	-0.0025	-0.0019	-0.0227	-0.1674	-0.0727	0.0082
200	-0.0360	0.0726	-0.0010	-0.0048	0.0455	-0.0900	-0.0724	0.0000
400	-0.0582	0.0314	-0.0009	-0.0003	0.0754	-0.0341	-0.0719	-0.0019
800	-0.0661	0.0143	0.0020	-0.0001	0.0874	-0.0144	-0.0730	-0.0015
n				$\beta_0 = 0.68\ (m = 0.04)$				
100	-0.0674	0.1130	0.0010	-0.0024	0.0937	-0.1456	-0.1575	-0.0085
200	-0.1179	0.0790	-0.0008	-0.0051	0.1532	-0.0949	-0.1602	-0.0012
400	-0.1341	0.0405	-0.0009	-0.0001	0.1747	-0.0457	-0.1576	0.0009
800	-0.1363	0.0180	0.0022	-0.0001	0.1802	-0.0191	-0.1554	-0.0005
n				$\beta_0 = 0.53\ (m = 0.06)$				
100	-0.0936	0.1214	-0.0068	-0.0030	0.1384	-0.1538	-0.2252	-0.0073
200	-0.1428	0.0919	-0.0055	-0.0047	0.2027	-0.1116	-0.2322	0.0028
400	-0.1925	0.0407	-0.0019	0.0003	0.2558	-0.0462	-0.2515	0.0013
800	-0.2094	0.0219	0.0013	0.0002	0.2744	-0.0243	-0.2582	0.0011
n				$\beta_0 = 0.41\ (m = 0.08)$				
100	-0.0508	0.1318	-0.0004	-0.0004	0.1286	-0.1660	-0.2550	-0.0050
200	-0.1114	0.0984	-0.0012	-0.0043	0.2061	-0.1204	-0.2758	0.0070
400	-0.1496	0.0500	0.0014	0.0005	0.2493	-0.0586	-0.2939	0.0060
800	-0.1681	0.0245	0.0190	0.0004	0.3277	-0.0281	-0.3193	0.0029

Note: All results are based on 500 replications. The true parameters in simulation are: $\xi = (\beta_0, \beta_1, \sigma_u, \sigma_v)^{\mathrm{T}} = \left(-F^{-1}(m),\ 0,\ 1,\ 0.6667\right)^{\mathrm{T}}$, where β_0 is chosen to ensure the probability that the dependent variable latent y_i^* falls below zero at a specified m under various configurations.

Table 3. Bias of the Censored (L_1) and Standard (L_0) MLE:

$\beta_0 = -F^{-1}(m),\ \beta_1 = 1,\ \sigma_v = 0.6667,\ \sigma_u = 1$

	β_0		β_1		σ_u		σ_v	
L	L_0	L_1	L_0	L_1	L_0	L_1	L_0	L_1
n	$\beta_0 = 0.91\ (m = 0.02)$							
100	-0.0114	0.0990	-0.1547	-0.0020	0.0536	-0.1258	-0.1622	-0.0201
200	-0.0201	0.0698	-0.1270	-0.0061	0.0790	-0.0830	-0.1444	-0.0072
400	-0.0060	0.0404	-0.1361	0.0003	0.0772	-0.0462	-0.1298	0.0019
800	-0.0160	0.0189	-0.1268	0.0006	0.0916	-0.0210	-0.1300	0.0012
n	$\beta_0 = 0.68\ (m = 0.04)$							
100	-0.0360	0.1089	-0.2179	-0.0040	0.1087	-0.1377	-0.2235	-0.0177
200	-0.0190	0.0757	-0.1804	-0.0065	0.1081	-0.0902	-0.1922	-0.0048
400	0.0004	0.0470	-0.1870	0.0003	0.1019	-0.0546	-0.1738	0.0052
800	-0.0100	0.0235	-0.1764	0.0005	0.1168	-0.0267	-0.1752	0.0025
n	$\beta_0 = 0.53\ (m = 0.06)$							
100	-0.0472	0.1298	-0.2735	-0.0023	0.1443	-0.1644	-0.2741	-0.0080
200	-0.0127	0.0803	-0.2251	-0.0060	0.1255	-0.0964	-0.2291	-0.0026
400	0.0082	0.0503	-0.2285	0.0002	0.1194	-0.0588	-0.2090	0.0055
800	-0.0010	0.0279	-0.2171	0.0005	0.1332	-0.0324	-0.2103	0.0042
n	$\beta_0 = 0.41\ (m = 0.08)$							
100	-0.0427	0.1342	-0.3201	-0.0018	0.1617	-0.1701	-0.3141	-0.0078
200	-0.0053	0.0895	-0.2686	-0.0057	0.1405	-0.1081	-0.2638	-0.0009
400	0.0166	0.0535	-0.2680	0.0000	0.1346	-0.0627	-0.2422	0.0055
800	0.0096	0.0274	-0.2557	-0.0002	0.1461	-0.0311	-0.2422	0.0030

Note: All results are based on 500 replications. The true parameters in simulation are: $\xi = (\beta_0, \beta_1, \sigma_u, \sigma_v)^{\mathrm{T}} = \left(-F^{-1}(m),\ 0,\ 1,\ 0.6667\right)^{\mathrm{T}}$, where β_0 is chosen to ensure the probability that the dependent variable latent y_i^* falls below zero at a specified m under various configurations.

Table 4. Mean Squared Errors (MSE) of the Censored (L_1) and Standard (L_0) MLE:

$\beta_0 = -F^{-1}(m),\ \beta_1 = 0,\ \sigma_v = 0.6667,\ \sigma_u = 1$

	β_0		β_1		σ_u		σ_v	
L	L_0	L_1	L_0	L_1	L_0	L_1	L_0	L_1
n				$\beta_0 = 0.91\ (m = 0.02)$				
100	0.0966	0.1523	0.0109	0.0114	0.1410	0.2305	0.0269	0.0243
200	0.0405	0.0834	0.0045	0.0049	0.0593	0.1270	0.0151	0.0135
400	0.0169	0.0341	0.0020	0.0021	0.0248	0.0532	0.0096	0.0069
800	0.0105	0.0124	0.0010	0.0011	0.0163	0.0182	0.0075	0.0035
n				$\beta_0 = 0.68\ (m = 0.04)$				
100	0.0852	0.1487	0.0122	0.0120	0.1409	0.2269	0.0497	0.0296
200	0.0459	0.0873	0.0041	0.0049	0.0661	0.1346	0.0371	0.0156
400	0.0309	0.0414	0.0018	0.0021	0.0468	0.0654	0.0300	0.0077
800	0.0244	0.0155	0.0009	0.0011	0.0399	0.0232	0.0264	0.0040
n				$\beta_0 = 0.53\ (m = 0.06)$				
100	0.0653	0.1507	0.0099	0.0115	0.0978	0.2298	0.0740	0.0313
200	0.0432	0.0971	0.0039	0.0049	0.0714	0.1517	0.0652	0.0174
400	0.0473	0.0409	0.0017	0.0021	0.0781	0.0648	0.0690	0.0081
800	0.0500	0.0183	0.0008	0.0011	0.0824	0.0278	0.0700	0.0046
n				$\beta_0 = 0.41\ (m = 0.08)$				
100	0.0431	0.1572	0.0094	0.0119	0.0759	0.2409	0.0858	0.0340
200	0.0278	0.0995	0.0059	0.0050	0.0811	0.1557	0.0855	0.0184
400	0.0337	0.0465	0.0034	0.0022	0.0960	0.0741	0.0923	0.0091
800	0.0432	0.0198	0.0207	0.0011	0.2547	0.0303	0.1067	0.0052

Note: All results are based on 500 replications. The true parameters in simulation are: $\xi = (\beta_0, \beta_1, \sigma_u, \sigma_v)^{\mathrm{T}} = \left(-F^{-1}(m),\ 0,\ 1,\ 0.6667\right)^{\mathrm{T}}$, where β_0 is chosen to ensure the probability that the dependent variable latent y_i^* falls below zero at a specified m under various configurations.

Table 5. Mean Squared Errors (MSE) of the Censored (L_1) and Standard (L_0) MLE:

$\beta_0 = -F^{-1}(m),\ \beta_1 = 1,\ \sigma_v = 0.6667,\ \sigma_u = 1$

	β_0		β_1		σ_u		σ_v	
L	L_0	L_1	L_0	L_1	L_0	L_1	L_0	L_1
n				$\beta_0 = 0.91\ (m = 0.02)$				
100	0.0771	0.1390	0.0328	0.0137	0.1134	0.2115	0.0508	0.0317
200	0.0276	0.0826	0.0201	0.0056	0.0471	0.1256	0.0307	0.0172
400	0.0105	0.0393	0.0202	0.0025	0.0220	0.0623	0.0217	0.0085
800	0.0053	0.0155	0.0169	0.0013	0.0158	0.0228	0.0193	0.0043
n				$\beta_0 = 0.68\ (m = 0.04)$				
100	0.0613	0.1429	0.0574	0.0140	0.0945	0.2170	0.0714	0.0343
200	0.0212	0.0844	0.0368	0.0059	0.0424	0.1295	0.0457	0.0180
400	0.0088	0.0426	0.0368	0.0026	0.0238	0.0676	0.0348	0.0091
800	0.0043	0.0186	0.0320	0.0013	0.0199	0.0281	0.0329	0.0048
n				$\beta_0 = 0.53\ (m = 0.06)$				
100	0.0529	0.1517	0.0874	0.0139	0.0879	0.2360	0.0952	0.0348
200	0.0182	0.0862	0.0555	0.0060	0.0419	0.1330	0.0607	0.0192
400	0.0079	0.0461	0.0542	0.0027	0.0261	0.0730	0.0480	0.0099
800	0.0037	0.0218	0.0481	0.0014	0.0232	0.0334	0.0464	0.0054
n				$\beta_0 = 0.41\ (m = 0.08)$				
100	0.0408	0.1556	0.1168	0.0147	0.0776	0.2401	0.1171	0.0374
200	0.0158	0.0933	0.0779	0.0063	0.0422	0.1437	0.0776	0.0205
400	0.0074	0.0491	0.0741	0.0028	0.0288	0.0780	0.0628	0.0107
800	0.0034	0.0213	0.0664	0.0014	0.0263	0.0327	0.0607	0.0056

Note: All results are based on 500 replications. The true parameters in simulation are: $\xi = (\beta_0, \beta_1, \sigma_u, \sigma_v)^{\mathrm{T}} = \left(-F^{-1}(m),\ 0,\ 1,\ 0.6667\right)^{\mathrm{T}}$, where β_0 is chosen to ensure the probability that the dependent variable latent y_i^* falls below zero at a specified m under various configurations.

Table 6. Mean Squared Errors (MSE) of the Censored (L_1) MLE: $\beta_0 = -F^{-1}(m)$, $\beta_1 = 0$, $\sigma_u = 1.0$

	β_0			β_1			σ_u			σ_v		
						$\lambda = 1.50$ $(\sigma_v \approx 0.6667)$						
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.0834	0.0873	0.0971	0.0049	0.0049	0.0049	0.1270	0.1346	0.1517	0.0135	0.0156	0.0174
400	0.0341	0.0414	0.0409	0.0021	0.0021	0.0021	0.0532	0.0654	0.0648	0.0069	0.0077	0.0081
800	0.0124	0.0155	0.0183	0.0011	0.0011	0.0011	0.0182	0.0232	0.0278	0.0035	0.0040	0.0046
						$\lambda = 1.25$ $(\sigma_v = 0.80)$						
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.1331	0.1399	0.1424	0.0060	0.0061	0.0061	0.2122	0.2250	0.2321	0.0171	0.0194	0.0211
400	0.0805	0.1028	0.1056	0.0027	0.0027	0.0027	0.1281	0.1678	0.1732	0.0108	0.0132	0.0143
800	0.0368	0.0485	0.0543	0.0013	0.0013	0.0013	0.0588	0.0786	0.0876	0.0056	0.0070	0.0080
						$\lambda = 0.85$ $(\sigma_v \approx 1.1765)$						
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.2617	0.2574	0.2635	0.0108	0.0109	0.0111	0.4119	0.4092	0.4204	0.0248	0.0276	0.0293
400	0.2250	0.2295	0.2427	0.0048	0.0048	0.0048	0.3592	0.3698	0.3939	0.0170	0.0187	0.0202
800	0.1852	0.1992	0.2146	0.0023	0.0023	0.0024	0.2929	0.3172	0.3424	0.0119	0.0135	0.0149

Note: All results are based on 500 replications. The true parameters in simulation are:

$\xi = (\beta_0, \beta_1, \sigma_u, \sigma_v)^{\mathrm{T}} = \left(-F^{-1}(m),\ 0,\ 1,\ \sigma_v\right)^{\mathrm{T}}$, where β_0 is chosen to ensure the probability that the dependent variable latent y_i^* falls below zero at a specified m under various configurations.

Table 7. Mean Squared Errors (MSE) of the Censored (L_1) MLE: $\beta_0 = -F^{-1}(m)$, $\beta_1 = 0$, $\sigma_u = 1.2$

	β_0			β_1			σ_u			σ_v		
						$\lambda = 1.50$ $(\sigma_v = 0.80)$						
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.1257	0.1353	0.1420	0.0068	0.0069	0.0069	0.2018	0.2192	0.2313	0.0211	0.0241	0.0261
400	0.0462	0.0598	0.0782	0.0031	0.0031	0.0031	0.0741	0.0982	0.1293	0.0106	0.0130	0.0154
800	0.0171	0.0223	0.0283	0.0016	0.0015	0.0015	0.0271	0.0356	0.0457	0.0051	0.0060	0.0070
						$\lambda = 1.25$ $(\sigma_v = 0.96)$						
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.1915	0.1960	0.2051	0.0087	0.0088	0.0088	0.3052	0.3164	0.3342	0.0246	0.0277	0.0303
400	0.1160	0.1499	0.1545	0.0039	0.0039	0.0039	0.1845	0.2444	0.2531	0.0155	0.0192	0.0208
800	0.0525	0.0692	0.0804	0.0019	0.0019	0.0019	0.0841	0.1118	0.1299	0.0080	0.0101	0.0117
						$\lambda = 0.85$ $(\sigma_v \approx 1.4118)$						
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.3729	0.3698	0.3762	0.0156	0.0157	0.0159	0.5852	0.5857	0.6001	0.0355	0.0397	0.0419
400	0.3241	0.3267	0.3481	0.0069	0.0069	0.0069	0.5172	0.5261	0.5641	0.0245	0.0269	0.0291
800	0.2673	0.2900	0.3056	0.0034	0.0034	0.0034	0.4227	0.4621	0.4866	0.0172	0.0197	0.0211

Note: All results are based on 500 replications. The true parameters in simulation are: $\xi = (\beta_0, \beta_1, \sigma_u, \sigma_v)^{\mathrm{T}} = (-F^{-1}(m),\ 0,\ 1,\ \sigma_v)^{\mathrm{T}}$, where β_0 is chosen to ensure the probability that the dependent variable latent y_i^* falls below zero at a specified m under various configurations.

Table 8. Mean Squared Errors (MSE) of the Censored (L_1) MLE: $\beta_0 = -F^{-1}(m)$, $\beta_1 = 0$, $\sigma_u = 1.4$

	β_0			β_1			σ_u			σ_v		
							$\lambda = 1.50$ $(\sigma_v \approx 0.9333)$					
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.1725	0.1822	0.1911	0.0093	0.0094	0.0094	0.2767	0.2955	0.3126	0.0288	0.0326	0.0355
400	0.0620	0.0796	0.1066	0.0042	0.0042	0.0042	0.0997	0.1309	0.1759	0.0144	0.0175	0.0210
800	0.0232	0.0302	0.0386	0.0021	0.0021	0.0021	0.0368	0.0481	0.0624	0.0069	0.0081	0.0096
							$\lambda = 1.25$ $(\sigma_v = 1.12)$					
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.2614	0.2740	0.2776	0.0119	0.0120	0.0120	0.4167	0.4422	0.4522	0.0336	0.0380	0.0411
400	0.1593	0.2099	0.2102	0.0053	0.0053	0.0053	0.2536	0.3420	0.3445	0.0212	0.0265	0.0283
800	0.0715	0.0923	0.1094	0.0026	0.0026	0.0026	0.1145	0.1490	0.1765	0.0109	0.0136	0.0160
							$\lambda = 0.85$ $(\sigma_v \approx 1.6471)$					
n	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06	m=0.02	m=0.04	m=0.06
200	0.5078	0.5033	0.5149	0.0212	0.0214	0.0217	0.7978	0.7980	0.8201	0.0484	0.0540	0.0572
400	0.4412	0.4483	0.4751	0.0093	0.0093	0.0094	0.7041	0.7221	0.7701	0.0333	0.0368	0.0399
800	0.3639	0.3900	0.4180	0.0046	0.0046	0.0046	0.5754	0.6220	0.6647	0.0234	0.0266	0.0289

Note: All results are based on 500 replications. The true parameters in simulation are: $\xi = \left(\beta_0, \beta_1, \sigma_u, \sigma_v\right)^{\mathrm{T}} = \left(-F^{-1}(m),\ 0,\ 1,\ \sigma_v\right)^{\mathrm{T}}$, where β_0 is chosen to ensure the probability that the dependent variable latent y_i^* falls below zero at a specified m under various configurations.

Table 9. Censored (L_1) and Standard (L_0) MLE of the Reservation Wages Regressions

Variable	Censored (L_1) MLE	Standard (L_0) MLE
Gov	0.3505 (0.0282)	0.3316 (0.0240)
Gender	–0.2846 (0.0221)	–0.2275 (0.0180)
Marry	0.0970 (0.0300)	0.1086 (0.0251)
Edu	0.5931 (0.0448)	0.4773 (0.0382)
Farming	–0.1850 (0.0483)	–0.1117 (0.0377)
Age	1.9093 (0.2887)	1.2933 (0.2252)
Age^2	–0.7823 (0.1396)	–0.5207 (0.1084)
Constant	8.1437 (0.1474)	8.5635 (0.1135)
σ_u	0.2961 (0.0428)	0.3631 (0.0244)
σ_v	0.2979 (0.0175)	0.2125 (0.0128)

Note: The number in parenthesis denotes the standard error.

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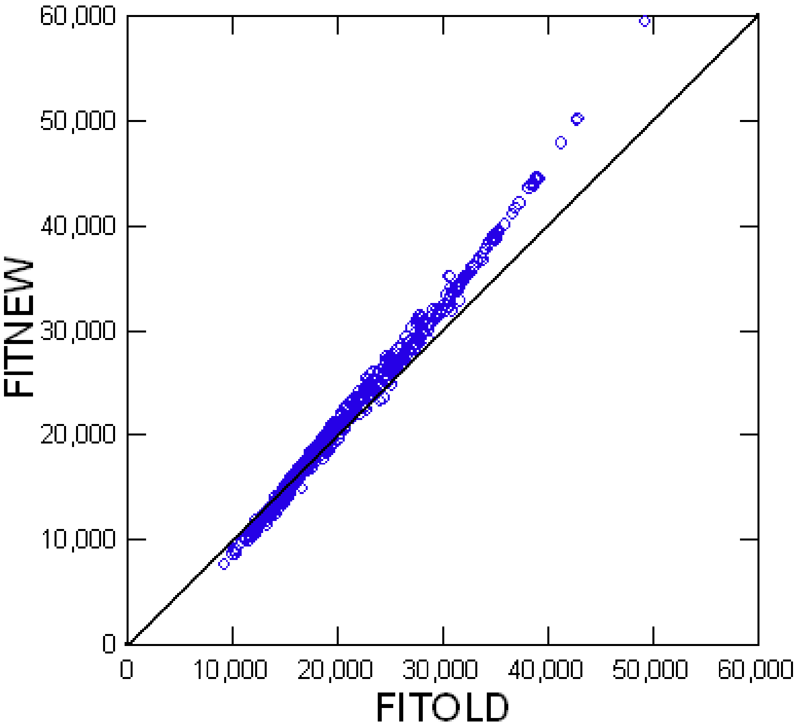
Figure 1: Predictions of Reservation Wages



Note: FITOLD is the predicted reservation wage from the standard SFA model, while FITNEW is the predicted reservation wage from the censored SFA approach.